PAST EXAM QUESTIONS AND SOLUTIONS

EEM 314 Signals and Systems Midterm Exam Sami Arıca

QUESTIONS May 11, 2002

1. Fourier series coefficients of a periodic signal $x[n]$ is given as

k	-2	-1	0	1	2
a_k	$-1.5-j1.5\sqrt{3}$	$3-j3\sqrt{3}$	3	$3+j3\sqrt{3}$	$-1.5+j1.5\sqrt{3}$

   Find signal $x[n]$.

2. Find convolution sum of the following signals.

$$h[n]=\begin{cases} 3n, & n=1,2,3,4 \\ 0, & elsewhere \end{cases},$$

$$x[n]=\begin{cases} 2, & n=-1,0,1,2,3,4,5 \\ 0 & elsewhere \end{cases}.$$

3. Find z-Transform of $x[n]=(n+1)\,a^n\cdot u[n]$. *Hint*: $\sum\limits_{n=0}^{\infty} nb^{n-1}=\frac{d}{db}\left(\sum\limits_{n=0}^{\infty} b^n\right)$.

4. Poles of system response of a causal and stable system are inside the unit circle. Prove this property. *Hint*: Remember that impulse response of a stable system is absolutely summable. You should also remember that ROC of a right-sided sequence is outside region of a circle.

5. Find z-Transform of $y[n]=x[2n]$ in terms of z-Transform of $x[n]$. *Hint*: Obtain $y[n]$ in two spets; I. $y_1[n]=\frac{1}{2}x[n]+\frac{1}{2}(-1)^n x[n]$, II. $y[n]=y_1[2n]$. First find z-Transform o $y_1[n]$ and later $y[n]$. (This question was not included in the exam questions)

Good Luck

EEM 314 Signals and Systems Final Exam

Sami Arıca

June 11, 2002

Instructions. Answer all questions.

QUESTIONS

1. Find z-transforms of the follwing signals.

   a) $x[n] = a^n u[n]$ b) $h[n] = u[n] * x[n] = \sum_{k=0}^{n} x[k]$.

2. Find inverse Laplace transforms of the following Laplace transform.

   $X(s) = \dfrac{1}{s-b} + \dfrac{1}{s+a}, \quad -a < \mathrm{Re}\{s\} < b\ , \quad -a < b\ .$

3. Find inverse z-transform of the following z-transform.

   $X(z) = \dfrac{1}{1-az^{-1}} + \dfrac{1}{1-bz^{-1}}, \quad |a| < |z| < |b|\ , \quad |a| < |b|\ .$

4. Response of an LTI system to input signal

$$x[n] = \begin{cases} n+1, & n = 0,1,2 \\ 0, & elsewhere \end{cases}$$

   is

$$y[n] = \begin{cases} (0.5)^n, & n = 0,1,2 \\ 0, & elsewhere \end{cases} .$$

   Find response of the system for input signal

$$x_1[n] = \begin{cases} n+1, & n = 0,1,2 \\ 4(n-2), & n = 3,4,5 \\ 0, & elsewhere \end{cases} .$$

Good Luck.

ANSWERS

1. a) $X(z) = \dfrac{1}{1 - az^{-1}}, \quad |z| > |a|$. b) $H(z) = \dfrac{1}{1 - z^{-1}} \cdot \dfrac{1}{1 - az^{-1}}, (|z| > |a|) \cap (|z| > 1)$.

2. $x(t) = e^{-at}u(t) - be^{bt}u(-t)$.

3. $x[n] = a^n u[n] - b^n u[-n-1]$.

4. $x_1[n] = x[n] + 4x[n-3]$,

$$y_1[n] = y[n] + 4y[n-3] = \begin{cases} (0.5)^n, & n = 0, 1, 2 \\ 32(0.5)^n, & n = 3, 4, 5 \\ 0, & elsewhere \end{cases} .$$

EEM 314 Signals and Systems Reset Exam

Sami Arıca

June 24, 2002

Instructions. Answer all questions.

QUESTIONS

1. For input $x[n] = u[n]$, output of an LTI system is given as $y[n] = 2\left(1 - (1/2)^{n+1}\right)u[n]$. Find impulse response of the system.

EEM 314 Signals and Systems II Midterm Exam I - April 04, 2003

QUESTIONS

Q1) $x(n) = u(n+1) - u(n-1)$ is input to an LTI system described by difference equation,

$$y(n) - \frac{1}{2}y(n-1) = x(n) \ .$$

Find output signal $x(n)$.

Q2)

$$x(n) = \begin{cases} \frac{1}{3} - \frac{1}{3}|n|, & n = -2, -1, 0, 1, 2 \\ 0, & elsewhere \end{cases}$$

Find $y(n) = x(n) * x(n)$. ($*$ is the convolution operator)

Q3)

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^{|n|}, & n = -2, -1, 0, 1, 2 \\ x(n + l \cdot 5), & l \in Z \end{cases} .$$

Find complex Fourier series coefficients of the signal.

Q4) Consider the following set of signals:

$$\begin{aligned} \phi(n) &= \left(\frac{1}{2}\right)^{|n|} u(n) \\ \phi_k(n) &= \phi(n-k), \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots \end{aligned}$$

Show that an arbitrary signal can be expressed in the form,

$$x(n) = \sum_{k=-\infty}^{\infty} a_k \phi_k(n) \ ,$$

by determining an explicit expression for the coefficient a_k in terms of the values of the signal $x(n)$. (Hint: what is the representation for $\delta(n)$?)

(This question is quoted from the textbook: Oppenheim and Willsky, Signals and Systems, Prentice/Hall International, Inc.)

ANSWERS

A1) Linear and causal systems can be described by constant coefficient difference equations. Then, because $x(n) = 0$ for $n \leq -2$,

$$\begin{aligned} n &\leq -2 \\ y(n) &= 0\ . \end{aligned}$$

The output samples for $n > -2$ are

$$\begin{aligned} n &= -1 \\ y(-1) &= \frac{1}{2}y(-2) + x(-1) \\ &= 1\ , \\ n &= 0 \\ y(0) &= \frac{1}{2}y(-1) + x(0) \\ &= \frac{1}{2} \cdot 1 + 1 \\ &= \frac{3}{2}\ , \\ n &= 1 \\ y(1) &= \frac{1}{2}y(0) + x(1) \\ &= \frac{1}{2} \cdot \frac{3}{2} + 0 \\ &= \frac{3}{4} \\ &\vdots \\ &\text{etc.} \end{aligned}$$

The general form of the output is,

$$y(n) = \left(\frac{1}{2}\right)^{n+1} u(n+1) + \left(\frac{1}{2}\right)^{n} u(n) \ .$$

$$\begin{aligned} x(n) &= u(n+1) - u(n-1) \\ &= \delta(n+1) + \delta(n) \end{aligned}$$

$\left(\frac{1}{2}\right)^{n} u(n)$ is response to $\delta(n)$.

A2)

$$\begin{aligned} y(n) &= \sum_{k=-2}^{2} x(k)\, x(n-k) \\ &= x(-2)\, x(n+2) + x(-1)\, x(n+1) + x(0)\, x(n) + x(1)\, x(n-1) + x(2)\, x(n+2) \ . \end{aligned}$$

$y(n)$ is zero if arguments of $x(n-k)$ and $x(k)$ are outside of the ranges,

$$-2 \geq n-k \leq 2$$

$$-2 \geq k \leq 2 \ .$$

Then, $y(n)$ should be calculated for $-4 \geq n \leq 4$. Because $x(n) = x(-n)$, $y(n) = y(-n)$. Finding $y(n)$ for $n = 0 \ldots 4$ will be sufficient.

$$
\begin{aligned}
y(0) &= 1\cdot 1+\frac{2}{3}\cdot\frac{2}{3}+\frac{1}{3}\cdot\frac{1}{3}+\frac{2}{3}\cdot\frac{2}{3}+1\cdot 1\\
&= 3\ ,\\
y(-1)=y(1) &= \frac{2}{3}\cdot 1+\frac{1}{3}\cdot\frac{2}{3}+\frac{2}{3}\cdot\frac{1}{3}+1\cdot\frac{2}{3}\\
&= \frac{16}{9}\ ,\\
y(-2)=y(2) &= \frac{1}{3}\cdot 1+\frac{2}{3}\cdot\frac{2}{3}+1\cdot\frac{1}{3}\\
&= 2\ ,\\
y(-3)=y(3) &= \frac{2}{3}\cdot 1+1\cdot\frac{2}{3}\\
&= \frac{4}{3}\ ,\\
y(-4)=y(4) &= 1\cdot 1\\
&= 1\ .
\end{aligned}
$$

A3) The period of signal, $x(n)$ is $N=5$. Because $x(n)\in R$ and $x(n)=x(-n)$, $a_k=a_{-k}$.

$$
\begin{aligned}
a_k &= \frac{1}{5}\cdot\sum_{n=-2}^{2} x(n)\, e^{-j\frac{2\pi}{5}n\cdot k}\\
&= \frac{1}{10}\left(2+2\cos\left(\frac{2\pi}{5}k\right)+\cos\left(\frac{4\pi}{5}k\right)\right)\ .
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= 0.5\\
a_{-1}=a_1 &= 0.18\\
a_{-2}=a_2 &= 0.07\ .
\end{aligned}
$$

A4) The representation of $\delta(n)$ in terms of $\phi(n)$ is,

$$\delta(n) = \phi(n) - \frac{1}{2}\phi(n-1) \quad .$$

First signal $x(n)$ is represented by impulses and next equivalent of impulse is replaced.

$$\begin{aligned}
x(n) &= \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \\
&= \sum_{k=-\infty}^{\infty} x(k)\left[\phi(n-k) - \frac{1}{2}\phi(n-k-1)\right] \\
&= \sum_{k=-\infty}^{\infty} x(k)\,\phi(n-k) - \frac{1}{2}\sum_{k=-\infty}^{\infty} x(k)\,\phi(n-k-1) \\
&= \sum_{k=-\infty}^{\infty} x(k)\,\phi(n-k) - \frac{1}{2}\sum_{k=-\infty}^{\infty} x(k+1)\,\phi(n-k) \\
&= \sum_{k=-\infty}^{\infty} \left[x(k) - \frac{1}{2}x(k+1)\right]\phi(n-k)
\end{aligned}$$

The coefficient a_k in terms of the values of $x(n)$ is,

$$a_k = x(k) - \frac{1}{2}x(k+1) \quad .$$

EEM 314 Signals and Systems II Midterm Exam II - May 2, 2003

QUESTIONS

Q1) Using the known z-transforms, find the inverse z-transform of,

$$X(z) = \frac{2+2z^{-1}}{(1+3z^{-1})(1-z^{-1})}, \quad 1 < |z| < 3 .$$

Q2) Find the Fourier transform of the following signals in terms of, $X(\Omega) = F[x(n)]$.

1. $y(n) = x^{\star}(n)$,
2. $y(n) = nx(n)$.

ANSWERS

A1) $X(z)$ is partitioned by using partial fraction expansion,

$$X(z) = \frac{1}{1+3z^{-1}} + \frac{1}{1-z^{-1}} \quad .$$

The parts of the z-transform are,

$$X_1(z) = \frac{1}{1+3z^{-1}}, \quad |z| < 3 \quad .$$

$$X_2(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \quad .$$

The inverse z-transforms of the parts are,

$$F^{-1}[X_1(z)] = (-3)^n u(-n-1) \quad .$$

$$F^{-1}[X_2(z)] = u(n) \quad .$$

Then,

$$F^{-1}[X(z)] = (-3)^n u(-n-1) + u(n) \quad .$$

A2)

1)

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x^{*}(n)\, e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(x(n)\, e^{j\Omega n}\right)^{*} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{j\Omega n}\right)^{*} \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j(-\Omega) n}\right)^{*} \\
&= X^{*}(-\Omega) \quad .
\end{aligned}$$

2)

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} n x(n)\, e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{\infty} x(n) \left[n e^{-j\Omega n}\right] \\
&= \sum_{n=-\infty}^{\infty} x(n) \left[j \frac{d}{d\Omega} \left(e^{-j\Omega n}\right)\right] \\
&= \sum_{n=-\infty}^{\infty} j \frac{d}{d\Omega} \left[x(n)\, e^{-j\Omega n}\right] \\
&= j \frac{d}{d\Omega} \left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\right] \\
&= j \frac{d}{d\Omega} X(\Omega) \quad .
\end{aligned}$$

Or

$$
\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \\
\frac{d}{d\Omega} X(\Omega) &= \frac{d}{d\Omega}\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}\right] \\
&= \sum_{n=-\infty}^{\infty} \frac{d}{d\Omega}\left[x(n)\, e^{-j\Omega n}\right] \\
&= \sum_{n=-\infty}^{\infty} x(n)\, \frac{d}{d\Omega}\left[e^{-j\Omega n}\right] \\
&= \sum_{n=-\infty}^{\infty} x(n)\left[-jne^{-j\Omega n}\right] \\
&= -j\sum_{n=-\infty}^{\infty} nx(n)\, e^{-j\Omega n} \\
&= -jY(\Omega) \quad .
\end{aligned}
$$

Inverse z-Transform

$$\frac{1}{2\pi j}\oint_{\Gamma} z^{n-1} dz = \delta(n)$$

$$\begin{aligned} & \frac{1}{2\pi j}\oint_{\Gamma} \left(x(k) z^{-k}\right) z^{n-1} dz \\ = & \frac{1}{2\pi j} x(k) \oint_{\Gamma} z^{n-k-1} dz \\ = & \delta(n-k) \cdot x(k) \end{aligned}$$

EEM 314 Signals and Systems II Final Exam - June 13, 2003

QUESTIONS

Q1) $x(n) = -2 \cdot \delta(n+2) + \delta(n-1)$ is input to an LTI system described by difference equation,

$$y(n) - \frac{1}{2} y(n-1) = 3 \cdot x(n) \ .$$

Find output signal, $y(n)$ for input signal, $x(n)$.

Q2)

$$x(n) = \begin{cases} 2^n, & n = -1, 0, 1, 2 \\ x(n + l \cdot 4), & l \in Z \end{cases} \ .$$

Find complex Fourier series coefficients of the signal.

Q3) Using the known z-transforms, find the inverse z-transform of,

$$X(z) = \frac{3z^{-1}}{2 - 5z^{-1} + 2z^{-2}} \ , \quad |z| > 2 \ .$$

Q4) Find Laplace transform of,

$$x(t) = 6^{-(2+3j)t} \, u(t) \ .$$

ANSWERS

A1)

Input		Output
$\delta(n)$	$\rightarrow$	$3\left(\frac{1}{2}\right)^{n} \cdot u(n)$
$\delta(n+2)$	$\rightarrow$	$-2 \cdot 3\left(\frac{1}{2}\right)^{n+2} \cdot u(n+2)$
$\delta(n-1)$	$\rightarrow$	$\left(\frac{1}{2}\right)^{n-1} \cdot u(n-1)$

$$y(n) = -2 \cdot 3\left(\frac{1}{2}\right)^{n+2} \cdot u(n+2) + \delta(n-1) + \left(\frac{1}{2}\right)^{n-1} \cdot u(n-1)$$

A2) Period of the signal, $N = 4$.

$$\begin{aligned} a_k &= \frac{1}{4}\left[\frac{1}{2}e^{j\frac{\pi}{2}k} + 1 + 2e^{-j\frac{\pi}{2}k} + 4e^{-j\pi k}\right] \\ & \frac{1}{4}\left[\frac{1}{2}(j)^k + 1 + 2(-j)^k + 4(-1)^k\right] \end{aligned}$$

A3)

$$X(z) = \frac{1}{1-2z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad (|z|>2) \bigcap \left(|z| > \frac{1}{2}\right)$$

$$x(n) = (2)^n u(n) - \left(\frac{1}{2}\right)^n u(n).$$

A4)

$$6 = e^{1.79}$$

$$x(t) = e^{-1.79(2+3j)t} \cdot u(t)$$

$$X(s) = \frac{1}{s+3.58+5.37j}, \quad \mathrm{Re}[s] > -3.58.$$

QUESTIONS

Q1) Find Fourier transform of

a)

$$x(t) = \begin{cases} -1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & elsewhere \end{cases}$$

b) $y(t) = \int\limits_{-\infty}^{t} x(\tau)\, d\tau$.

Q2) Find inverse Laplace transform of

$$X(s) = \frac{1}{(s+2)(4s+1)^2} \quad \mathrm{Re}\{s\} > -\frac{1}{4}$$

using partial fraction expansion method.

Q3)

$$x(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & elsewhere \end{cases} .$$

a) Find energy of the signal.

b) Find and plot $y(t) = -2x(2-3t) + 2x(3t-1)$.

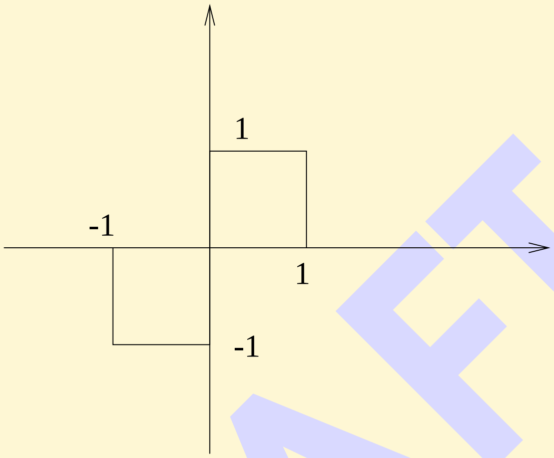
Q4) Find trigonometric Fourier series coefficients of $x(t) = |\cos(\pi t)|$.

Good Luck

ANSWERS

A1)

a)



$$
\begin{aligned}
X(\omega) &= -\int_{-1}^{0} e^{-j\omega t}dt + \int_{0}^{1} e^{-j\omega t}dt \\
&= -\int_{0}^{1} e^{j\omega t}dt + \int_{0}^{1} e^{-j\omega t}dt \\
&= -2j\int_{0}^{1} \sin(\omega t)\,dt \\
&= 2j\frac{1}{\omega}\cos(\omega t)\Big|_{0}^{1} \\
&= 2j\frac{1}{\omega}\left(\cos(\omega) - 1\right)
\end{aligned}
$$

b)

$$\begin{aligned} Y(\omega) &= \frac{1}{j\omega}X(\omega) + \pi X(0)\,\delta(\omega) \\ &= \frac{2(\cos(\omega)-1)}{\omega^2} \end{aligned}$$

A2)

$$\begin{aligned} X(s) &= \frac{1/16}{(s+2)(s+1/4)^2} \\ &= \frac{A}{(s+2)} + \frac{B}{(s+1/4)} + \frac{C}{(s+1/4)^2} \end{aligned}$$

$$\begin{aligned} A &= \lim_{s \to -2} (s+2)\, X(s) \\ &= 1/49 \end{aligned}$$

$$\begin{aligned} C &= \lim_{s \to -1/4} (s+1/4)^2\, X(s) \\ &= 1/28 \end{aligned}$$

$$\begin{aligned} B &= \lim_{s \to -1/4} \frac{d}{ds} (s+1/4)^2\, X(s) \\ &= \lim_{s \to -1/4} -\frac{1/16}{(s+2)^2} \\ &= -1/49 \end{aligned}$$

$$\begin{aligned}\frac{dY(s)}{ds} &\leftrightarrow -ty(t)\\ -\frac{1}{(s+1/4)} &\leftrightarrow -e^{-t/4}u(t)\\ \frac{1}{(s+1/4)^2} &\leftrightarrow te^{-t/4}u(t)\end{aligned}$$

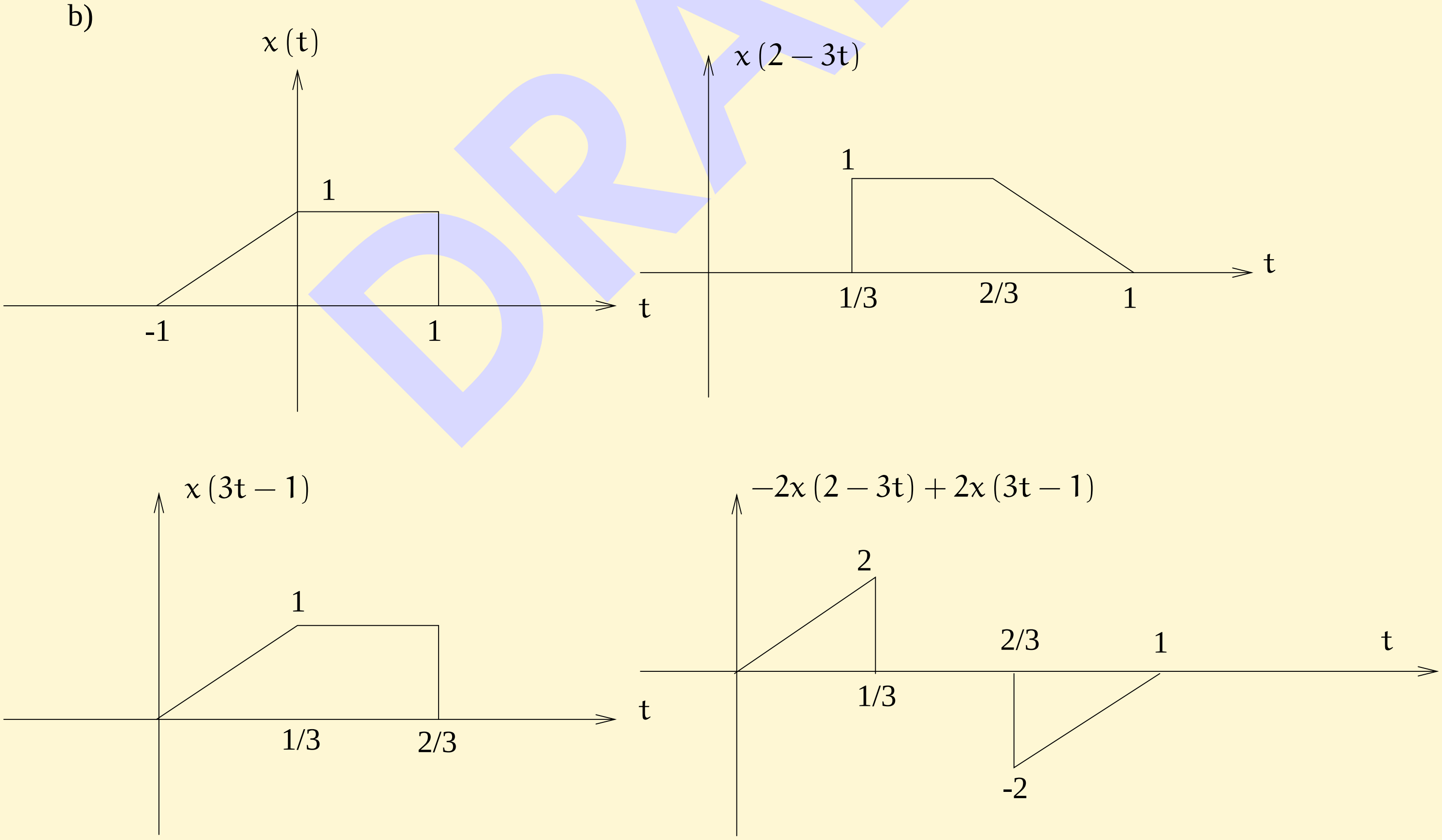
$$x(t) = 1/49\ e^{-2t}u(t) - 1/49\ e^{-t/4}u(t) + 1/28\ te^{-t/4}u(t)$$

A3)

a)

$$
\begin{aligned}
& \int_{-1}^{0} (t+1)^2 \, dt + \int_{-1}^{0} (1)^2 \, dt \\
= & \int_{0}^{1} (u)^2 \, du + 1 \\
= & \left. \frac{1}{3} u^3 \right|_{0}^{1} + 1 \\
= & \frac{4}{3}
\end{aligned}
$$

b)



A4)

$$
\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1/2}^{1/2} \cos(\pi t)\,dt \\
&= \frac{1}{\pi}\sin(\pi t)\Big|_{-1/2}^{1/2} \\
&= \frac{1}{\pi}\left(\sin(\pi/2) + \sin(\pi/2)\right) \\
&= \frac{2}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{2}{1}\int_{-1/2}^{1/2} \cos(\pi t)\cdot\cos(2\pi k t)\,dt \\
&= 2\int_{-1/2}^{1/2} \frac{1}{2}\left[\cos((1+2k)\pi t) + \cos((1-2k)\pi t)\right] dt \\
&= \frac{2}{\pi(1+2k)}\sin\left(\frac{\pi}{2}(1+2k)\right) + \frac{2}{\pi(1-2k)}\sin\left(\frac{\pi}{2}(1-2k)\right)
\end{aligned}
$$

$$
\begin{aligned}
b_k &= \frac{2}{1}\int_{-1/2}^{1/2} \cos(\pi t)\cdot\sin(2\pi k t)\,dt \\
&= \int_{-1/2}^{1/2} \sin((1+2k)\pi t)\,dt + \int_{-1/2}^{1/2} \sin((-1+2k)\pi t)\,dt \\
&= 0
\end{aligned}
$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(2\pi k t)$$

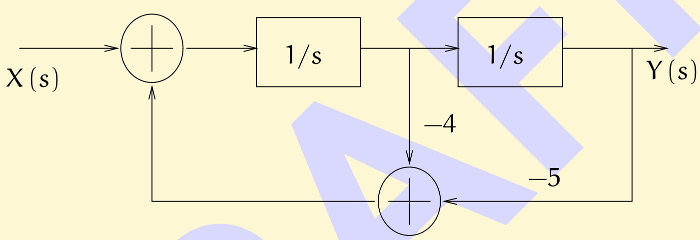
QUESTIONS

Q1) Fourier transform of signal $x(t)$ is given as,

$$X(\omega) = \frac{2(1-\cos(\omega))}{\omega^2} \quad .$$

Find Fourier transform of $x(0.5t-2)$. ($\begin{array}{c} x(t) \leftrightarrow X(\omega) \\ x(0.5t-2) \leftrightarrow ? \end{array}$)

Q2)



a) Find system response $H(s)$ of the system. b) Find differential equation which describes the input-output relationship of the system.

Q3) Find Fourier series coefficients of

$$x[n] = (n-2) \mod 5 \quad .$$

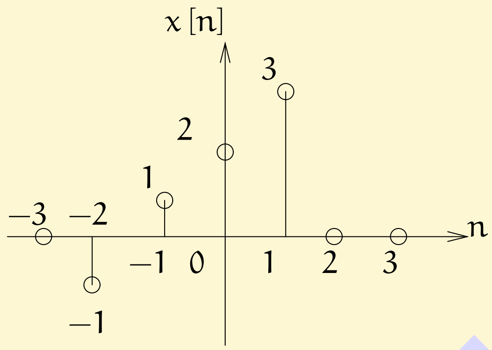
Here, $x \mod y$ is the reminder of integer division and is always positive. $13 \mod 5 = 3$, $-9 \mod 5 = 1$.

Q4)

$$X(z) = -21\frac{z}{(5z-2)(2z-5)}, \quad \frac{2}{5} < |z| < \frac{5}{2}.$$

Find inverse z-transform using partial fraction expansion method.

Q5)



$y_1[n] = x[2n]$ and $y_2[n] = \begin{cases} x[n/2], & n = 0, \mp 2, \mp 4, \mp 6, \ldots \\ 0, & \text{otherwise} \end{cases}$.

a) Find and plot $y_1[n]$ and $y_2[n]$. b) Find and plot $y[n] = 3y_1[-n] - 2y_2[n]$.

(Take plot range as $n = -6 \ldots 6$) .

Q6) Find Laplace transform of $x(t) = e^{-2|t|}$.

The following questions are given as exercises.

Q7) An LTI system is described by the following differential equation.

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = x(t) \quad .$$

Initial conditions for $x(t) = u(t)$ are given as, $y(0) = 0$, and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$. Fint unit step response , $s(t)$, of the system.

Q8) There is many definitions of Dirac delta function. One of these is,

$$\delta(x) = \lim_{a \to 0} \int_{-\infty}^{\infty} e^{j2\pi x(y - a|y|)} dy \quad .$$

Let $\phi(w, t) = e^{j2\pi\omega t}$ is basis function of Fourier transform pairs ;

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,\phi(w,t)\,dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,\phi^{*}(w,t)\,d\omega \quad .$$

Using definition of Dirac delta given above, prove that $x(t)$ can be reconstructed from $X(\omega)$.

ANSWERS

A1)

$$Y(\omega) = \int_{-\infty}^{\infty} x(0.5t-2)\, e^{-j\omega t} dt \quad ,$$

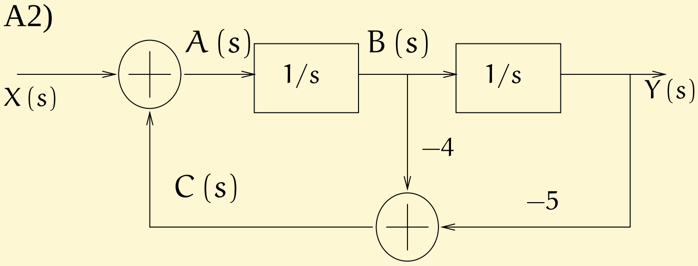
changing integral variable t with $u = 0.5t - 2$,

$$\begin{aligned} t &= 2u + 4 \\ dt &= 2du \\ t = -\infty &\Rightarrow u = -\infty \\ t = \infty &\Rightarrow u = \infty \quad , \end{aligned}$$

$Y(\omega)$ becomes,

$$\begin{aligned} Y(\omega) &= 2\int_{-\infty}^{\infty} x(u)\, e^{-j\omega(2u+4)} du \\ &= 2e^{-j4\omega} \int_{-\infty}^{\infty} x(u)\, e^{-j2\omega u} du \\ &= 2e^{-j4\omega} X(2\omega) \\ &= e^{-j4\omega} \frac{(1-\cos(2\omega))}{\omega^2} \quad . \end{aligned}$$

A2)



$$\begin{aligned} A(s) &= X(s) + C(s) \\ C(s) &= -5Y(s) - 4B(s) \\ A(s) &= s^2 Y(s) \\ B(s) &= sY(s) \quad . \end{aligned}$$

Combining the equations we obtain,

$$s^2 Y(s) = X(s) - 5Y(s) - 4sY(s)$$
$$s^2 Y(s) + 4sY(s) + 5Y(s) = X(s) \quad .$$

The system response is,

$$\begin{aligned} H(s) &= \frac{Y(s)}{X(s)} \\ &= \frac{1}{s^2 + 4s + 5} \quad . \end{aligned}$$

Time domain equivalent of the system is,

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = x(t) \quad .$$

A3) Period of $x[n]$ is $N = 5$.

$$\begin{array}{cccccc} n & 0 & 1 & 2 & 3 & 4 \\ x[n] & 3 & 4 & 0 & 1 & 2 \end{array} .$$

$$\begin{aligned}a_k &= \frac{1}{5}\sum_{n=0}^{4} x[n]\, e^{-j\frac{2\pi k}{5}} \\ &= \frac{1}{5}\left(3+4e^{-j\frac{2\pi}{5}}+e^{-j\frac{6\pi}{5}}+2e^{-j\frac{8\pi k}{5}}\right) \\ &= \frac{1}{5}\left(3+4e^{-j\frac{2\pi}{5}}+e^{j\frac{4\pi}{5}}+2e^{j\frac{2\pi k}{5}}\right)\end{aligned}$$

The Fourier series coefficients,

k	0	1	2	3	4
a_k	2	$0.81-0.26j$	$-0.31-0.43j$	$-0.31+0.43j$	$0.81+0.26j$

A4)

$$\begin{aligned}\frac{X(z)}{z} &= -\frac{21}{(5z-2)(2z-5)} \\ &= \frac{5}{5z-2}-\frac{2}{2z-2}\ .\end{aligned}$$

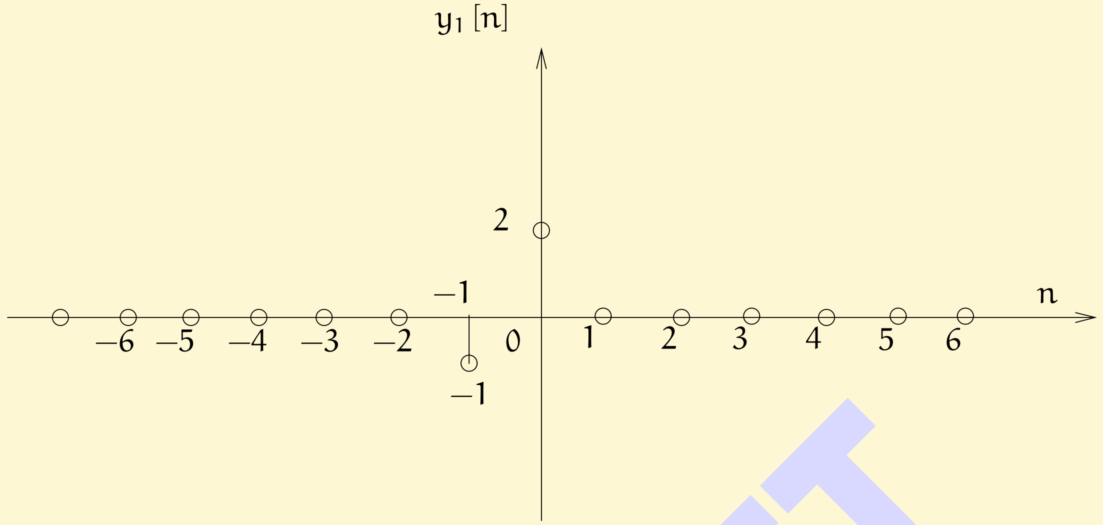
$$\begin{aligned}X(z) &= \frac{5z}{5z-2}-\frac{2z}{2z-2} \\ &= \frac{1}{1-\frac{2}{5}z^{-1}}-\frac{1}{1-\frac{5}{2}z^{-1}} \\ &= A(z)+B(z)\ .\end{aligned}$$

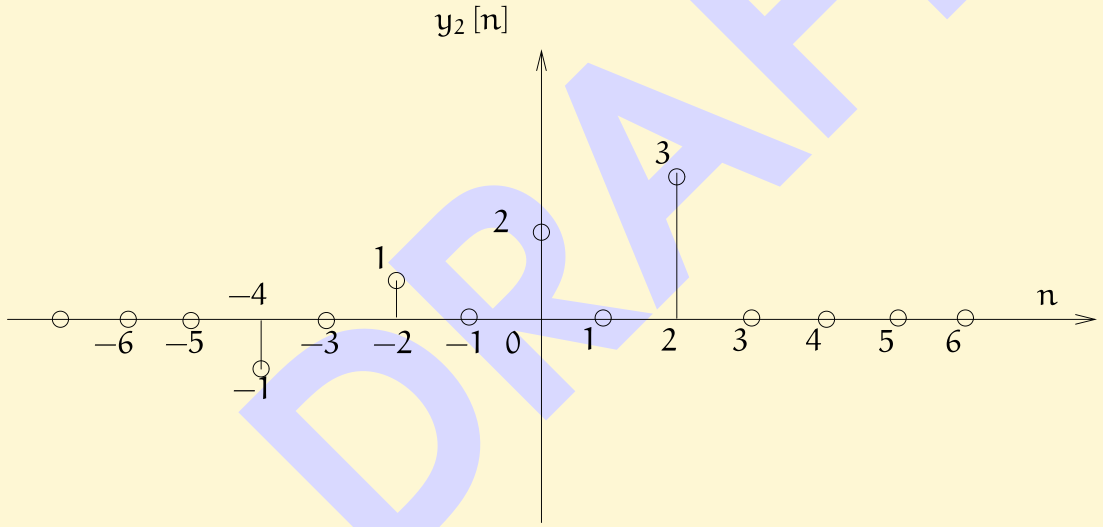
$$\begin{aligned} A(z) &= \frac{1}{1-\frac{2}{5}z^{-1}}, \quad |z| > \frac{2}{5} \\ B(z) &= \frac{-1}{1-\frac{5}{2}z^{-1}}, \quad |z| < \frac{5}{2} \ . \end{aligned}$$

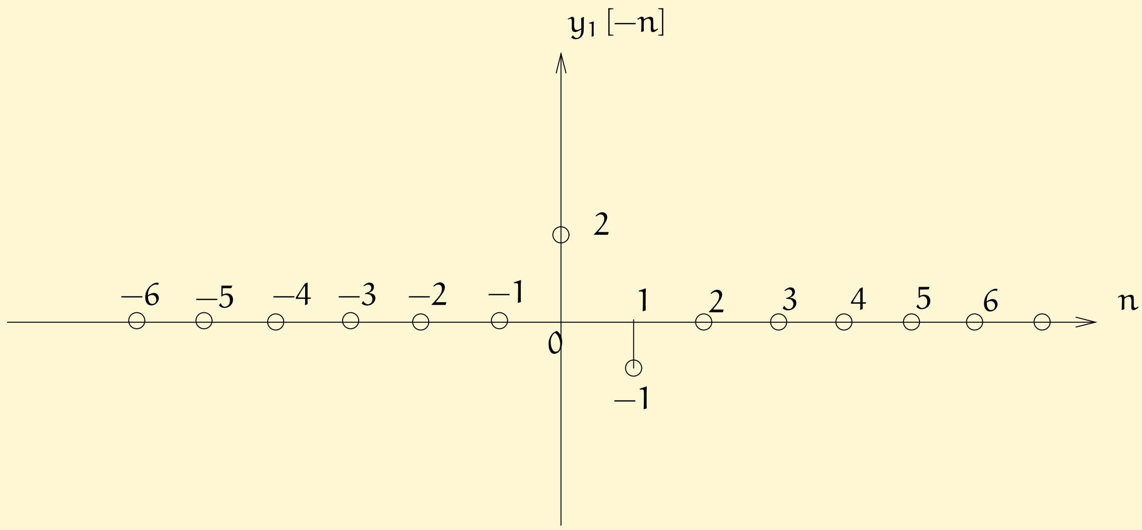
$$\begin{aligned} a[n] &= \left(\frac{2}{5}\right)^n u[n] \ , \\ b[n] &= \left(\frac{5}{2}\right)^n u[-n-1] \ . \end{aligned}$$

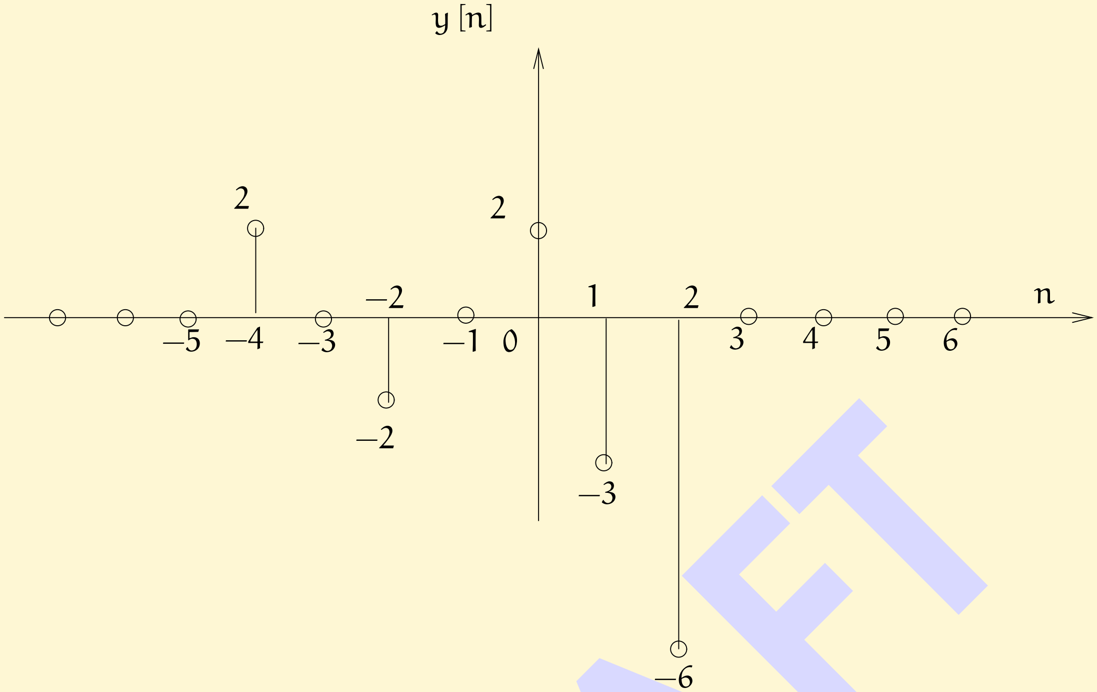
$$\begin{aligned} x[n] &= a[n] + b[n] \\ &= \left(\frac{2}{5}\right)^n u[n] + \left(\frac{5}{2}\right)^n u[-n-1] \ . \end{aligned}$$

A5)


$y_1[n]$
2
−1
n
−6
−5
−4
−3
−2
0
1
2
3
4
5
6
−1


$y_2[n]$
3
2
1
−4
n
−6
−5
−3
−2
−1
0
1
2
3
4
5
6
−1


$y_1[-n]$
2
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
n
0
−1



A6)

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} e^{-2|t|} e^{-st} dt \\
&= \int_{-\infty}^{0} e^{2t} e^{-st} dt + \int_{0}^{\infty} e^{-2t} e^{-st} dt \\
&= \int_{0}^{\infty} e^{-2t} e^{st} dt + \int_{0}^{\infty} e^{-2t} e^{-st} dt
\end{aligned}
$$

$$
\begin{aligned}
\int_{0}^{\infty} e^{(s-2)t} dt &= -\frac{1}{s-2}, \quad \operatorname{Re}\{s\} < 2 \\
\int_{0}^{\infty} e^{-(s+2)t} dt &= \frac{1}{s+2}, \quad \operatorname{Re}\{s\} > -2
\end{aligned}
$$

$$\begin{aligned} X(s) &= -\frac{1}{s-2} + \frac{1}{s+2} \\ &= \frac{4}{4-s^2}, \quad -2 < \mathrm{Re}\{s\} < 2 \end{aligned}$$

QUESTIONS

Q1) An LTI system is described by the following differential equation.

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+5y(t)=x(t) \quad .$$

Find unit step response , $s(t)$, of the system.

Q2) System function of a continuous-time LTI system is given as,

$$H(s)=\frac{1}{s+1}\cdot\frac{2}{2s+1}$$

Using the bilinear transform, $s=\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}}$, find discrete time equivalent ($H(z)$) of the system function and write difference equation of the discrete-time system (the sampling period $T=2$).

Q3) Poles of transfer function of a discrete-time LTI system is given as, $0.5+j0.25$, $0.5-j0.25$, $0.25+j0.5$, $0.25-j0.5$ and zeros of the transfer function are $1.5+j0.25$ and $1.5-j0.25$. $H(0)=1$. Find the transfer function and frequency response of the system.

Q4) Plot cascade form realization of continuous-time system with system function $H(s)$ given in Q2.

Q5) Find trigonometric Fourier series coefficients of $x(t)=3\cdot|\cos(\pi t-\pi/4)|$.

ANSWERS

A1) Since this differential equation represents an LTI system $y(t) = 0$ for $t < 0$. Then a solution for $t \geq 0$ should be looked for. The characteristic equation of the differential equation is,

$$s^2 + 4\,s + 5 = 0\,.$$

The roots are $-2 + j$ and $-2 - j$. Then the homogenous solution is,

$$y_h(t) = a\,e^{(-2+j)t} + b\,e^{(-2-j)t}\,.$$

Because input is a constant, the particular solution is a zero degree polynomial;

$$y_p(t) = 1/5\,.$$

The complete solution is,

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= a\,e^{(-2+j)t} + b\,e^{(-2-j)t} + 1/5\,. \end{aligned}$$

The inition conditions, $y(0) = 0$ and $\left.\dfrac{dy(t)}{dt}\right|_{t=0} = 0$ are used to find unknown constants, a and b. They lead to following equations,

$$\begin{aligned} a + b &= -1/5 \\ (-2+j)\,a + (-2-j)\,b &= 0\,. \end{aligned}$$

Solution for these equations is $a = -1/10 + 1/5\,j$ and $b = -1/10 - 1/5\,j$. Then,

$$\begin{aligned} y(t) &= (-1/10 + 1/5\,j)\, e^{(-2+j)t} + (-1/10 - 1/5\,j)\, e^{(-2-j)t} + 1/5 \\ &= 1/5 - 2/5\, e^{-2t} \sin(t) - 1/5\, e^{-2t} \cos(t)\, . \end{aligned}$$

As a result the unit step response the LTI sytem;

$$s(t) = \left(1/5 - 2/5\, e^{-2t} \sin(t) - 1/5\, e^{-2t} \cos(t)\right) u(t)\ .$$

A2)

$$\begin{aligned} H(z) &= H(s)|_{s=(1-z^{-1})/(1+z^{-1})} \\ &= \frac{(z+1)^2}{z\,(3z-1)} \\ &= \frac{1 + 2\,z^{-1} + z^{-2}}{3 - z^{-1}} \end{aligned}$$

The input output relation in z-domain,

$$Y(z) = H(z) \cdot X(z)\ .$$

The corresponding difference equation;

$$3\,y(n) - y(n-1) = x(n) + 2\,x(n-1) + x(n-2)\ .$$

A3) The structure of transfer function of an LTI system is a rational polynomial function;

$$\begin{aligned} H(z) &= A\frac{1+b_1z^{-1}+b_2z^{-2}+\cdots+b_Mz^{-M}}{1+a_1z^(-1)+a_2z^{-2}+\cdots+a_Nz^{-N}} \\ &= A\frac{\left(1-r_1z^{-1}\right)\left(1-r_2z^{-1}\right)\cdots\left(1-r_Mz^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)\cdots\left(1-p_Nz^{-1}\right)}, \end{aligned}$$

where p_k and r_k are poles and zeros respectively. N is the order of the LTI system. Hence, for given poles and zeros the transfer function;

$$H(z) = \frac{A\cdot\left(1-(1.5+0.25\,j)\,z^{-1}\right)\left(1-(1.5-0.25\,j)\,z^{-1}\right)}{\left(1-(0.5+0.25\,j)\,z^{-1}\right)\left(1-(0.5-0.25\,j)\,z^{-1}\right)\left(1-(0.25+0.5\,j)\,z^{-1}\right)\left(1-(0.25-0.5\,j)\,z^{-1}\right)}.$$

Using $H(0)=1$, the unknown constant is found as, $A=0.0422$. Then the transfer function becomes,

$$H(z) = \frac{0.0422-0.1266\,z^{-1}+0.1055\,z^{-2}}{1-1.5\,z^{-1}+1.125\,z^{-2}-0.4688\,z^{-3}+0.09766\,z^{-4}}.$$

A4)

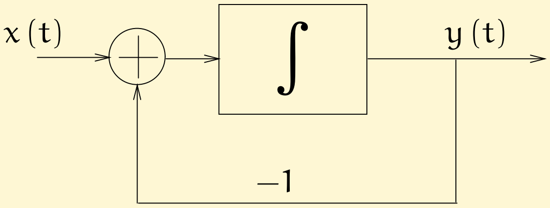
$$H(s) = H_1(s)\,H_2(s)\;..$$

$$\begin{aligned} H_1(s) &= \frac{1}{s+1}, \\ H_2(s) &= \frac{2}{2\,s+1}. \end{aligned}$$

I. Realization of $H_1(s)$;

$$Y(s) = H_1(s)\,X(s)\;.$$

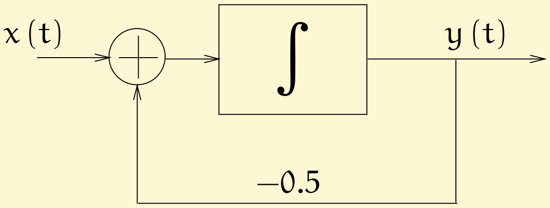
$$
\begin{aligned}
Y(s) &= \frac{1}{s+1}X(s)\ , \\
s\,Y(s)+Y(s) &= X(s)\ , \\
Y(s) &= \frac{1}{s}\left(X(s)-Y(s)\right)\ .
\end{aligned}
$$



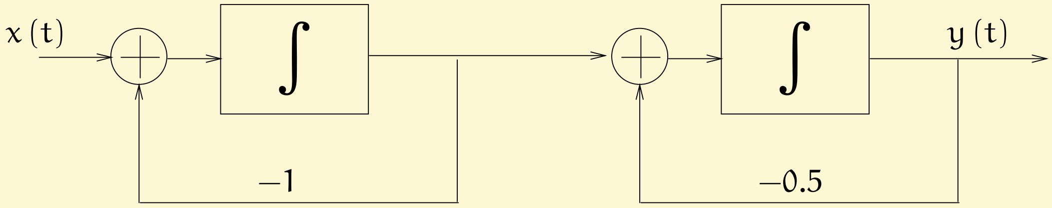
II. Realization of $H_2(s)$;

$$Y(s) = H_2(s)\,X(s)\ ..$$

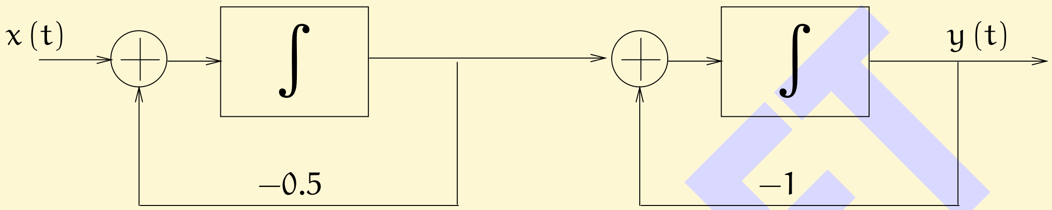
$$
\begin{aligned}
Y(s) &= \frac{2}{2\,s+1}X(s)\ , \\
2\,s\,Y(s)+Y(s) &= 2\,X(s)\ , \\
Y(s) &= \frac{1}{s}\left(X(s)-0.5\cdot Y(s)\right)\ .
\end{aligned}
$$



III. Realization of $H(s)$;



or,



A5) Period of $\cos(\pi t - \pi/4)$ is $2\ sec$. Period of the absolute value of the cosine decreases by half and becomes $1\ sec$. Then $T = 1\ sec$.

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt \\ &= \frac{1}{1}\int_{-1/4}^{3/4} \cos(\pi t - \pi/4)\,dt \\ &= \frac{2}{\pi}, \end{aligned}$$

$$
\begin{aligned}
a_k &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos\left(\frac{2\pi k t}{T}\right)dt \\
&= \frac{2}{1}\int_{-1/4}^{3/4} \cos(\pi t-\pi/4)\cos(2\pi k t)\,dt \\
&= -4\,\frac{\cos(\pi k/2)}{\pi\,(-1+4k^2)}\,,
\end{aligned}
$$

$$
\begin{aligned}
b_k &= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin\left(\frac{2\pi k t}{T}\right)dt \\
&= \frac{2}{1}\int_{-1/4}^{3/4} \cos(\pi t-\pi/4)\sin(2\pi k t)\,dt \\
&= 4\,\frac{\sin(\pi k/2)}{\pi\,(-1+4k^2)}\,.
\end{aligned}
$$

$$
x(t) = \frac{2}{\pi} - 4\sum_{k=1}^{\infty}\frac{\cos(\pi k/2)}{\pi\,(-1+4k^2)}\cos(2\pi k t) + 4\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{\pi\,(-1+4k^2)}\sin(2\pi k t)\ .
$$

QUESTIONS

Q1a) $x(t) = (1 - 2|t|)(u(t + 0.5) - u(t - 0.5))$ is given. Find and plot $y(t) = x(0.5t - 0.5) - x(0.5t - 1) - x(0.5t + 0.5) + x(0.5t + 1)$.

Q1b) $x[n] = \frac{1}{2}\delta[n + 2] + \delta[n] + 2\delta[n - 2]$ is given. Find and plot $y[n] = x[2n - 2] - x[2n - 4] - x[2n + 2] + x[2n + 4]$.

Q2a) A continuous-time LTI system is described by the following first order differential equation.

$$\frac{d}{dt}y(t) + 2y(t) = x(t) \quad .$$

Find i) unit step response, ii) impulse response, of the system and iii) response of the system to an arbitrary input.

Q2b) A discrete-time LTI system is described by the following first order difference equation.

$$y[n] - 0.2y[n - 1] = x[n] \quad .$$

Find i) unit step response, ii) impulse response, of the system and iii) response of the system to an arbitrary input.

Q3) A periodic signal is given as,

$$x(t) = \begin{cases} \sin(2\pi t), & -0.5 \le t \le 0.5 \\ 0, & -1 \le t < -0.5 \ \text{ and } \ 0.5 < t \le 1 \\ x(t + l \cdot 2), & l \in \mathbb{Z} \end{cases}$$

Find a) complex Fourier series representation, b) trigonometric Fourier series representation of the signal.

Q4a) Find state-space representation of $\frac{d^2}{dt^2}y(t) + 0.5\frac{d}{dt}y(t) + y(t) = x(t)$. Choose state variables as $z_1(t) = y(t)$ and $z_2(t) = \frac{d}{dt}y(t)$.

Q4b) Find state-space representation of $y[n] + 0.5y[n-1] + y[n-2] = x[n]$. Choose state variables as $z_1[n] = y[n-2]$ and $z_2[n] = y[n-1]$.

ANSWERS

A1a)

$$
\begin{aligned}
x(t) &= \begin{cases} 1+2t, & -1/2 < t \leq 0 \\ 1-2t, & 0 \leq t < 1/2 \\ 0, & \text{otherwise} \end{cases} \\
x(0.5t) &= \begin{cases} 1+t, & -1 < t \leq 0 \\ 1-t, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases} \\
x(0.5t-0.5) &= \begin{cases} t, & 0 < t \leq 1 \\ 2-t, & 1 \leq t < 2 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$-x(0.5t-1) = \begin{cases} 1-t, & 1 < t \leq 2 \\ t-3, & 2 \leq t < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$-x(0.5t+0.5) = \begin{cases} -t-2, & -2 < t \leq -1 \\ t, & -1 \leq t < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x(0.5t+1) = \begin{cases} 3+t, & -3 < t \leq -2 \\ -1-t, & -2 \leq t < -1 \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \begin{cases} 3+t, & -3 < t \leq -2 \\ -3-2t, & -2 < t \leq -1 \\ t, & -1 < t \leq 1 \\ 3-2t, & 1 < t \leq 2 \\ t-3, & 2 < t \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

A1b)

$$\begin{aligned}
x[n] &= \frac{1}{2}\delta[n+2] + \delta[n] + 2\delta[n-2] \\
x[2n-2] &= \frac{1}{2}\delta[2n] + \delta[2n-2] + 2\delta[2n-4] \\
-x[2n-4] &= -\frac{1}{2}\delta[2n-2] - \delta[2n-4] - 2\delta[2n-6] \\
-x[2n+2] &= -\frac{1}{2}\delta[2n+4] - \delta[2n+2] - 2\delta[2n] \\
x[2n+4] &= \frac{1}{2}\delta[2n+6] + \delta[2n+4] + 2\delta[2n+2]
\end{aligned}$$

$$\begin{aligned}
y[n] &= \frac{1}{2}\delta[2n+6] + \frac{1}{2}\delta[2n+4] + \frac{3}{2}\delta[2n+2] - \frac{3}{2}\delta[2n] + \frac{1}{2}\delta[2n-2] \\
&\quad +\delta[2n-4] - 2\delta[2n-6]
\end{aligned}$$

A2a)

$$\frac{d}{dt}y(t) + 2y(t) = x(t)\ .$$

To find impulse response first find the step response. The solution of the differential equation consists of homogenous and particular solutions. The homogenous solution is solution of the differential equation for $x(t) = 0$.

$$\frac{d}{dt}y_h(t) + 2y_h(t) = 0\ .$$

The homogenous solution is $y_h(t) = A \cdot e^{st}$ and s is found by solving the characteristic equation $s + 2 = 0$. ($\frac{d^k}{dt^k}$ is replaced by s^k). Hence

$$y_h(t) = A \cdot e^{-2t}\ .$$

The particular solution is solution of the differential equation

$$\frac{d}{dt}y_p(t) + 2y_p(t) = 1, \quad t \geq 0 \, .$$

Since the left hand side of the equation is constant the solution should be a constant, $y_p(t) = B$. From the differential equation,

$$\frac{d}{dt}B + 2B = 1, \quad t \geq 0 \, .$$

$B = 1/2$. Then the solution is $y(t) = y_h(t) + y_p(t) = A \cdot e^{-2t} + Bu(t)$. The initial condition $y(0) = 0$. The constant A is found from the initial value.

$$\begin{aligned} A + 1/2 &= 0 \\ A &= -1/2 \, . \end{aligned}$$

Hence,

$$y(t) = \begin{cases} \frac{1}{2}\left(1 - e^{-2t}\right), & t \geq 0 \\ -\frac{1}{2}e^{-2t}, & t < 0 \, . \end{cases}$$

Since the systems is linear it should be initially rest. The complete solution or response to unit step is,

$$\begin{aligned} s(t) &= y(t) \\ &= \frac{1}{2}\left(1 - e^{-2t}\right)u(t) \, . \end{aligned}$$

The impulse response of the system is simply derivative of the unit step response,

$$\begin{aligned} h(t) &= \frac{ds(t)}{dt} \\ &= e^{-2t}u(t) \ . \end{aligned}$$

The response to an arbitrary input is just convolution between input and the impulse response,

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\, e^{-2(t-\tau)} u(t-\tau) d\tau \\ &= e^{-2t} \int_{-\infty}^{t} x(\tau)\, e^{2\tau} d\tau \end{aligned}$$

A2b)

First solve the difference equation for $x[n] = u[n]$.

$$y[n] = 0.2y[n-1] + 1, \quad n \geq 0 \ .$$

$$
\begin{aligned}
y[-1] &= 0 \\
y[0] &= 0.2y[-1] + 1 = 1 \\
y[1] &= 0.2y[0] + 1 = 1 + 0.2 \\
y[2] &= 0.2y[1] + 1 = 0.2 + 0.2^2 \\
y[3] &= 0.2y[2] + 1 = ((1 + 0.2) + 0.2^2)0.2 + 1 = 1 + 0.2 + 0.2^2 + 0.2^3 \\
\ldots \\
y[n] &= \sum_{k=0}^{n} (0.2)^k = \frac{1 - (0.2)^{n+1}}{1 - 0.2} = \frac{1 - (0.2)^{n+1}}{0.8}
\end{aligned}
$$

Hence, the step response is

$$s[n] = \frac{1 - (0.2)^{n+1}}{0.8} u[n] \ .$$

The impulse response,

$$
\begin{aligned}
h[n] &= s[n] - s[n-1] \\
&= 0.2^n u(n) \ .
\end{aligned}
$$

The response to an arbitrary input is just convolution between input and the impulse response,

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\
&= \sum_{k=-\infty}^{\infty} x[k]\, (0.2)^{n-k} u[n-k] \\
&= (0.2)^n \sum_{k=-\infty}^{n} x[k]\, (0.2)^{-k}
\end{aligned}
$$

A3a)

$$
\begin{aligned}
a_k &= \frac{1}{2}\int_{-1/2}^{1/2} \sin(2\pi t)\, e^{-j\left(\frac{2\pi}{2}\right)kt} dt \\
&= \frac{1}{2}\int_{-1/2}^{0} \sin(2\pi t)\, e^{-j\pi kt} dt + \frac{1}{2}\int_{0}^{1/2} \sin(2\pi t)\, e^{-j\pi kt} dt \\
&= -\frac{1}{2}\int_{0}^{1/2} \sin(2\pi t)\, e^{j\pi kt} dt + \frac{1}{2}\int_{0}^{1/2} \sin(2\pi t)\, e^{-j\pi kt} dt \\
&= -j\int_{0}^{1/2} \sin(2\pi t)\sin(\pi kt)\, dt
\end{aligned}
$$

$$
\sin(2\pi t)\sin(\pi kt) = -\frac{1}{2}\cos(\pi(2+k)t) + \frac{1}{2}\cos(\pi(2-k)t)
$$

$$
\begin{aligned}
a_k &= j\int_{0}^{1/2} \cos(\pi(2+k)t)\, dt - j\int_{0}^{1/2} \cos(\pi(2-k)t)\, dt \\
&= j\frac{1}{2}\frac{1}{\pi(2+k)}\sin(\pi(2+k)t)\Big|_{0}^{1/2} - j\frac{1}{2}\frac{1}{\pi(2-k)}\sin(\pi(2+k)t)\Big|_{0}^{1/2} \\
&= \frac{1}{2j\pi(2+k)}\sin\left(\frac{\pi}{2}(2+k)\right) - \frac{1}{2j\pi(2-k)}\sin\left(\frac{\pi}{2}(2+k)\right)
\end{aligned}
$$

typesetting of the following needs to be corrected

$$
\begin{aligned}
a_k &= j\frac{1}{2}\frac{1}{\pi(2+k)}\sin(\pi(2+k)t)\Big|_0^{1/2} - j\frac{1}{2}\frac{1}{\pi(2-k)}\sin(\pi(2+k)t)\Big|_0^{1/2} \\
&= \frac{1}{2j\pi(2+k)}\sin\left(\frac{\pi}{2}(2+k)\right) - \frac{1}{2j\pi(2-k)}\sin\left(\frac{\pi}{2}(2+k)\right) \\
&= \frac{1}{2j\pi}\left[\frac{1}{2+k}\sin\left(k\frac{\pi}{2}\right) + \frac{1}{2-k}\sin\left(k\frac{\pi}{2}\right)\right] \\
&= \frac{\sin\left(k\frac{\pi}{2}\right)}{2j\pi}\left[\frac{4}{4-k^2}\right]
\end{aligned}
$$

$$x(t) = \sum_{k=\infty}^{\infty} \frac{2}{j\pi}\frac{\sin\left(k\frac{\pi}{2}\right)}{4-k^2} e^{j\pi kt}$$

A3b)

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left[a_k e^{-j\frac{2\pi}{T_0}kt}\right]$$

$$a_0 = 0$$

$$\mathrm{Re}\left[a_k e^{-j\frac{2\pi}{T_0}kt}\right] = -\frac{1}{2\pi}\sin\left(k\frac{\pi}{2}\frac{4}{4-k^2}\right)$$

$$x(t) = \frac{-1}{\pi}\sum_{k=1}^{\infty}\frac{4}{4-k^2}\sin\left(k\frac{\pi}{2}\right)\sin(k\pi t)$$

A4a)

$$\frac{d^2y(t)}{dt^2} + 0.5\frac{dy(t)}{dt} = x(t)$$
$$z_1(t) = y(t)$$
$$z_2(t) = \frac{dy(t)}{dt}$$
$$z_2(t) = \frac{dz_1(t)}{dt}$$
$$\frac{dz_2(t)}{dt} = -0.5z_2(t) - z_1(t) + x(t)$$
$$\begin{bmatrix} \frac{dz_1(t)}{dt} \\ \frac{dz_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} + 0x(t)$$

A4b)

$$y[n] + 0.5y[n-1] + y[n-2] = x[n]$$
$$z_1[n] = y[n-1]$$
$$z_2[n] = y[n-2]$$
$$z_2[n] = z_1[n-1]$$
$$z_2[n+1] = z_1[n]$$
$$y[n] = x[n] - 0.5z_1[n] - z_2[n]$$
$$z_1[n+1] = -0.5z_1[n] - z_2[n] + x[n]$$

$$\begin{bmatrix} z_1[n+1] \\ z_2[n+1] \end{bmatrix} = \begin{bmatrix} -0.5 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1[n] \\ z_2[n] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y[n] = \begin{bmatrix} -0.5 & -1 \end{bmatrix} \begin{bmatrix} z_1[n] \\ z_2[n] \end{bmatrix} + 1x[n]$$

QUESTIONS

Q1) Find Fourier transform of the following signal,

$$x(t) = \begin{cases} \cos(\pi t), & -0.5 \leq t \leq 0.5 \\ 0, & \text{otherwise} \end{cases}$$

Q2) Find Fourier transform of the following signal,

$$x(t) = \begin{cases} \cos(\pi t/2)\sin(4\pi t), & -1 \leq t \leq 1 \\ 0, & \textit{otherwise} \end{cases}$$

Q2) Find frequency response of the following system. And draw block diagram realization of the system.

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

Q3) Find inverse Fourier transform of,

$$X(\omega) = 4\frac{\sin^2(\omega)}{\omega^2} \quad .$$

(Use the convolution property of Fourier transforms.)

Q4) Find complex and trigonometric Fourier series coefficients of, a) periodic convolution of

$\sin(2\pi t)$ with itself, and b) periodic convolution of

$$x(t) = \begin{cases} 0, & -1/2 \leq t \leq 0 \\ 1, & 0 \leq t \leq 1/2 \\ x(t+l), & l \in \mathbb{Z} \end{cases}$$

with itself. (Use the periodic convolution property of Fourier series ($c_k = T_0 a_k b_k$).)

QUESTIONS

Q1a) $x(t) = (1 - |t|)(u(t+1) - u(t-1))$ is given. Find and plot $y(t) = x(2t-2) - x(2t-4) - x(2t+1) + x(2t+2)$.

Q1b) $x[n] = \frac{1}{2}\delta[n+2] + \delta[n] + 2\delta[n-2]$ is given. Find and plot $y[n] = x[n/2-1] - x[n/2-2] - x[n/2+1] + x[n/2+2]$.

Q2a) A continuous-time LTI system is described by the following first order differential equation.

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = 2x(t) \quad .$$

Find i) unit step response, ii) impulse response, of the system and iii) response of the system to an arbitrary input.

Q2b) A discrete-time LTI system is described by the following first order difference equation.

$$y[n] - 0.7y[n-1] + 0.1y[n-2] = 2x[n] - 0.7x[n-1] \quad .$$

Find i) unit step response, ii) impulse response, of the system and iii) response of the system to an arbitrary input.

Q3) A periodic signal is given as,

$$x(t) = \begin{cases} \sin\left(\frac{\pi}{2}t\right), & -1 \le t \le 1 \\ x(t + l \cdot 2), & l \in \mathbb{Z} \end{cases}$$

Find a) complex Fourier series representation, b) trigonometric Fourier series representation of the signal.

Q4a) Find state-space representation of $\frac{d^2}{dt^2}y(t) + 0.5\frac{d}{dt}y(t) + y(t) = x(t)$. Choose state variables as $z_1(t) = y(t)$ and $z_2(t) = \frac{d}{dt}y(t)$.

Q4b) Find state-space representation of $y[n] + 0.5y[n-1] + y[n-2] = x[n]$. Choose state variables as $z_1[n] = y[n-1]$ and $z_2[n] = y[n-1] - y[n-2]$.

QUESTIONS

Q1) Find Fourier transform of the following signal,

$$x(t) = \begin{cases} \cos(\pi t/2)\sin(4\pi t), & -1 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Q2) Find frequency response of the following system. And draw block diagram realization of the system.

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{d^2 x(t)}{dt^2} + 2\frac{dx(t)}{dt} + x(t)$$

Q3) Find inverse Fourier transform of,

$$X(\omega) = 2\pi e^{-2j\omega}\left(u(\omega+1) - u(\omega-1)\right) \quad .$$

Q4) Let $x(t)$ and $y(t)$ are periodic signals with period of T_0. Prove that complex Fourier series coefficients of $x(t)\,y(t)$ is,

$$x(t)\,y(t) \xrightarrow{F} \sum_{l=-\infty}^{\infty} a_l b_{k-l} = a_k * b_k \quad .$$

where a_k are b_k are Fourier series coefficients of $x(t)$ and $y(t)$ respectively.

QUESTIONS

Q1) A continuous-time LTI system is described by the following first order differential equation.

$$\frac{d}{dt}y(t) + 2y(t) = 2x(t) \quad .$$

Find a) unit step response, b) impulse response.

Q2) A periodic signal is given as,

$$x(t) = \begin{cases} \sin(4\pi t) + \cos(4\pi t), & -0.5 \le t \le 0.5 \\ 0, & -1 \le t < -0.5 \text{ and } 0.5 < t \le 1 \\ x(t + l \cdot 2), & l \in \mathbb{Z} \end{cases}$$

Find trigonometric Fourier series representation (sine and cosine form) of the signal.

Q3) Find state-space representation of $\frac{d^2}{dt^2}(t) + 0.5\frac{d}{dt}y(t) + y(t) = x(t)$. *Choose state variables as* $z_1(t) = y(t)$ *and* $z_2(t) = y(t) + \frac{d}{dt}y(t)$.

Q3) Find inverse Fourier transform of,

$$X(\omega) = 4\, e^{-j\omega}\, \frac{\sin^2(\omega)}{\omega^2} \quad .$$

(Use the time shift and the convolution property of Fourier transforms).

Q4) System function $(H(s))$of an LTI causal and stable system have poles at $s = -1 + j$ and $s = -1 - j$ and one zero at $s = -1$. $h(t)$ is the impulse response of the system. $h(0^+) = 2$

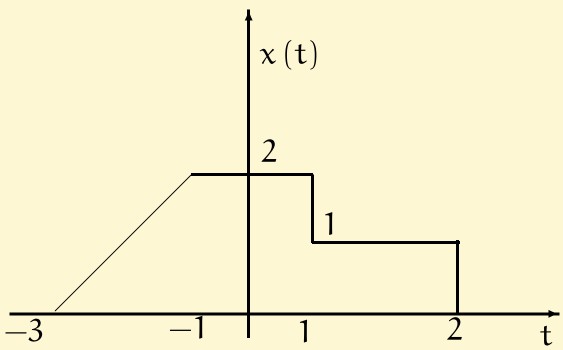
and $\left.\frac{dh(t)}{dt}\right|_{t=0^+} = -2$ are given. a) Find the system function and mark pole-zero locations and specify the region of convergence on the s-plane. b) Find the impulse response of the system.

Answer the first 4 questions. Q5, Q6, Q7 are left to you as exercises.

Exam time is 90 minutes.

QUESTIONS

Q1)



Find, a) $3x(-2t+3)$, b) $\int_{-\infty}^{\infty} x(t)\,dt$, c) $\int_{-\infty}^{\infty} x(2t)\,dt$, d) $\int_{-\infty}^{\infty} x^2(t)\,dt$, e) $\int_{-\infty}^{\infty} x^2(2t)\,dt$.

Q2) Find complex Fourier series coefficients of $x(t)$, $(T_o \geq T)$.

$$\tilde{x}(t) = \begin{cases} A\cos^2\left(\frac{\pi}{T}t\right), & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

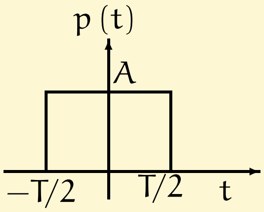
$$x(t) = \begin{cases} \tilde{x}(t), & -\frac{T_o}{2} \leq t \leq \frac{T_o}{2} \\ x(t + l \cdot T_o), & l \in Z \end{cases}$$

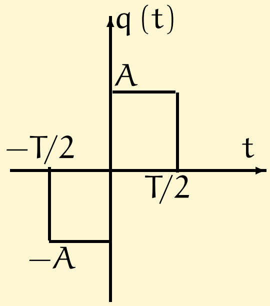
Q3) Fourier series coefficients of a real and periodic signal for $k \geq 0$ is given as;

k	0	1	2	3	$4\ldots\infty$
a_k	1	$2-j$	$3+4j$	$1-2j$	0

The period is, $T_o = 1\ sec$. Construct signal from its Fourier series.

Q4)





Find and plot convolution of $p(t)$ with $q(t)$.

Q5) Find Fourier transform of

a) $\delta(t)$,

b)

$$\begin{aligned} \frac{d}{dt}\left(a(t)\cdot b(t)\right) &= a(t)\cdot\frac{db(t)}{dt} + \frac{da(t)}{dt}\cdot b(t) \\ F[a(t)] &= A(\omega) \\ F[b(t)] &= B(\omega) \end{aligned}$$

c) $x(t) = e^{-at}u(t), \quad a > 0,$

d) $y(t) = \dfrac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0.$

Hint :

$$\begin{aligned} a(t) &= e^{-at} \\ b(t) &= u(t) \end{aligned}$$

Q6) Find Fourier transform of

a) $x(t) = e^{-a|t|}\left(u(t) - \frac{1}{2}\right), \quad a > 0.$

b) $y(t) = \lim_{a \to 0} x(t)$

Q7) Find complex Fourier series coefficients of $x(t)$.

$$\tilde{x}(t) = \begin{cases} p(t) + jq(t), & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & otherwise \end{cases}$$

$$x(t) = \begin{cases} \tilde{x}(t), & \\ x(t + l \cdot T_o), & l \in Z \end{cases}$$

where $p(t)$ and $q(t)$ are as in Q4.

(Since $x(-t) = x^*(t)$, the Fourier series coefficients are real !).

Q8) The magnitude,

$$A(\omega) = \begin{cases} A_o, & -2\pi B \le \omega \le 2\pi B \\ 0, & \text{otherwise} \end{cases}$$

and phase,

$$\theta(\omega) = \begin{cases} -\pi/2, & \omega \le 0 \\ \pi/2, & \omega > 0 \\ 0, & \text{otherwise} \end{cases}$$

of a Fourier transform are given. Find the inverse Fourier transform ($X(\omega) = A(\omega)\, e^{j\theta(\omega)}$).

Q9) Find the bilateral Laplace transform of

a) $\delta(t)$,

b)

$$
\begin{aligned}
\frac{d}{dt}\left(a(t)\cdot b(t)\right) &= a(t)\cdot\frac{db(t)}{dt}+\frac{da(t)}{dt}\cdot b(t) \\
L\left[a(t)\right] &= A(s) \\
L\left[b(t)\right] &= B(s)
\end{aligned}
$$

c) $x(t) = e^{-at}u(t), \quad a > 0,$

d) $y(t) = \dfrac{d}{dt}\left(e^{-at}u(t)\right), \quad a > 0.$

Hint :

$$
\begin{aligned}
a(t) &= e^{-at} \\
b(t) &= u(t)
\end{aligned}
$$

Q10) Find the unilateral Laplace transform of

a) $x(t) = e^{-at}, \quad a > 0,$

b) $y(t) = \dfrac{d}{dt}x(t).$

Reminding :

$$\mathcal{Y}(s) = s\,\mathcal{X}(s) - x(0)$$

This property can be easily proved by employing partial integration method.

ANSWERS

A1)

a) $x(-3t+3) = x(-(3(t-1)))$. The transformation of independent variable can be done in three steps.

$$\begin{aligned} x_a(t) &= x(-t) \\ x_b(t) &= x_a(3t) \\ y(t) &= 3x_b(t-1) \end{aligned}$$

Alternatively,

$$x(t) = \begin{cases} t+3, & -3 \le t < -1 \\ 2, & -1 \le t < 1 \\ 1, & 1 \le t \le 2 \end{cases}$$

$$\begin{aligned} y(t) &= \begin{cases} 3(-3t+3)+9, & -3 \le -3t+3 < -1 \\ 6, & -1 \le -3t+3 < 1 \\ 3, & 1 \le -3t+3 \le 2 \end{cases} \\ &= \begin{cases} -9t+18, & 2 \ge t > \frac{4}{3} \\ 6, & \frac{4}{3} \ge t > \frac{2}{3} \\ 3, & \frac{2}{3} \ge t \ge \frac{1}{3} \end{cases} \\ &= \begin{cases} 3, & \frac{1}{3} \le t \le \frac{2}{3} \\ 6, & \frac{2}{3} < t \le \frac{4}{3} \\ -9t+18, & \frac{4}{3} < t \le 2 \end{cases} \end{aligned}$$

b)

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)\,dt &= \int_{-3}^{-1} x(t)\,dt + \int_{-1}^{1} x(t)\,dt + \int_{1}^{2} x(t)\,dt \\ &= \frac{1}{2}2\cdot 2 + 2\cdot 2 + 1\cdot 1 \\ &= 7\end{aligned}$$

c)

$$\begin{aligned}\int_{-\infty}^{\infty} x(2t)\,dt &= \frac{1}{2}\int_{-\infty}^{\infty} x(t)\,dt \\ &= \frac{7}{2}\end{aligned}$$

d)

$$\begin{aligned}\int_{-\infty}^{\infty} x^2(t)\,dt &= \int_{-3}^{-1} x^2(t)\,dt + \int_{-1}^{1} x^2(t)\,dt + \int_{1}^{2} x^2(t)\,dt \\ &= \frac{1}{2}2\cdot 2^2 + 2\cdot 2^2 + 1\cdot 1^2 \\ &= 13\end{aligned}$$

e)

$$\begin{aligned}\int_{-\infty}^{\infty} x^2(2t)\,dt &= \frac{1}{2}\int_{-\infty}^{\infty} x^2(t)\,dt \\ &= \frac{13}{2}\end{aligned}$$

A2)

$$
\begin{aligned}
a_k &= \frac{1}{T_o}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos^2\left(\frac{\pi}{T}t\right)\, e^{-j\frac{2\pi}{T_o}kt}\,dt \\
&= \frac{2}{T_o}\int_{0}^{\frac{T}{2}} A\cos^2\left(\frac{\pi}{T}t\right)\cos\left(\frac{2\pi}{T_o}kt\right)dt \\
&= \frac{A}{T_o}\int_{0}^{\frac{T}{2}} \cos\left(\frac{2\pi}{T}t\right)dt + \frac{A}{T_o}\int_{0}^{\frac{T}{2}} \cos\left(\frac{2\pi}{T}t\right)\cos\left(\frac{2\pi}{T_o}kt\right)dt \\
&= \frac{A}{T_o}\int_{0}^{\frac{T}{2}} \cos\left(\frac{2\pi}{T}t\right)dt + \frac{A}{2T_o}\int_{0}^{\frac{T}{2}} \cos\left(2\pi\left(\frac{1}{T}-\frac{k}{T_o}\right)t\right)dt \\
&\quad + \frac{A}{2T_o}\int_{0}^{\frac{T}{2}} \cos\left(2\pi\left(\frac{1}{T}+\frac{k}{T_o}\right)t\right)dt \\
&= 0 - \frac{AT\sin\left(\frac{T\pi k}{To}\right)}{4\pi\left(-To+kT\right)} - \frac{AT\sin\left(\frac{T\pi k}{To}\right)}{4\pi\left(To+kT\right)} \\
&= -\frac{AT^2k}{2\pi}\,\frac{\sin\left(\frac{T\pi k}{To}\right)}{k^2T^2-T_o^2}
\end{aligned}
$$

A3)

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k\, e^{j\frac{2\pi}{T_o}kt} \\
&= 1 + (2+j)\, e^{-j\frac{2\pi}{T_o}t} + (2-j)\, e^{j\frac{2\pi}{T_o}t} + (3-4j)\, e^{-j\frac{2\pi}{T_o}2t} \\
&\quad + (3+4j)\, e^{j\frac{2\pi}{T_o}2t} + (1+2j)\, e^{-j\frac{2\pi}{T_o}3t} + (1-2j)\, e^{j\frac{2\pi}{T_o}3t} \\
&= 1 + 4\cos(2\pi t) + 2\sin(2\pi t) + 6\cos(4\pi t) \\
&\quad -8\sin(4\pi t) + 2\cos(6\pi t) + 4\sin(6\pi t)
\end{aligned}
$$

A4) Suppose that,

$$
g(t) = \begin{cases} A, & -\dfrac{T}{4} \le t \le \dfrac{T}{4} \\ 0, & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
a(t) &= g(t) * g(t) \\
&= \begin{cases} A^2\dfrac{T}{2}\left(1 - \dfrac{2}{T}|t|\right), & -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

The signals in terms of $g(t)$ are,

$$
\begin{aligned}
p(t) &= g\left(t + \frac{T}{4}\right) + g\left(t - \frac{T}{4}\right) \\
q(t) &= -g\left(t + \frac{T}{4}\right) + g\left(t - \frac{T}{4}\right)
\end{aligned}
$$

$$\begin{aligned} p(t) * q(t) &= \left[g\left(t+\frac{T}{4}\right)+g\left(t-\frac{T}{4}\right)\right] * \left[-g\left(t+\frac{T}{4}\right)+g\left(t-\frac{T}{4}\right)\right] \\ &= -g\left(t+\frac{T}{4}\right) * g\left(t+\frac{T}{4}\right)+g\left(t+\frac{T}{4}\right) * g\left(t-\frac{T}{4}\right) \\ &\quad -g\left(t-\frac{T}{4}\right) * g\left(t+\frac{T}{4}\right)+g\left(t-\frac{T}{4}\right) * g\left(t-\frac{T}{4}\right) \\ &= -a\left(t+\frac{T}{2}\right)+a(t)-a(t)+a\left(t-\frac{T}{2}\right) \\ &= -a\left(t+\frac{T}{2}\right)+a\left(t-\frac{T}{2}\right) \end{aligned}$$

A6)

a)

$$X(\omega)=\frac{j\omega}{(j\omega)^2-a^2}$$

b)

$$Y(\omega)=\frac{1}{j\omega}$$

A8)

$$x(t)=\frac{A_o}{\pi t}\left(\cos(2\pi B t)-1\right)$$

QUESTIONS

Q1) $x(t) = (1 - |t|)(u(t+1) - u(t-1))$ is given. Find and plot $y(t) = x(2t-2) - x(2t-4) - x(2t+1) + x(2t+2)$.

Q2) A continuous-time LTI system is described by the following first order differential equation.

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = 2x(t) \quad .$$

Find i) unit step response, ii) impulse response, of the system and iii) response of the system to an arbitrary input.

Q3) A periodic signal is given as,

$$x(t) = \begin{cases} \sin\left(\frac{\pi}{2}t\right), & -1 \le t \le 1 \\ x(t + l \cdot 2), & l \in \mathbb{Z} \end{cases}$$

Find a) complex Fourier series representation, b) trigonometric Fourier series representation of the signal.

Q4) Find inverse Fourier transform of,

$$X(\omega) = 2\pi e^{-2j\omega}(u(\omega+1) - u(\omega-1)) \quad .$$

Answer all questions. Exam time is 90 minutes.

QUESTIONS

Q1) Two discrete-time signals are given:

n	$a[n]$	$b[n]$
-2	1	0
-1	-2	-4
0	3	3
1	-1	3
2	1	-1
3	0	2
elsewhere	0	0

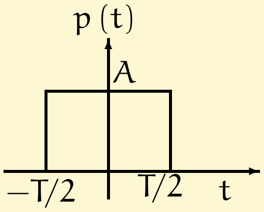
Find convolution, $c[n] = a[n] * b[n]$.

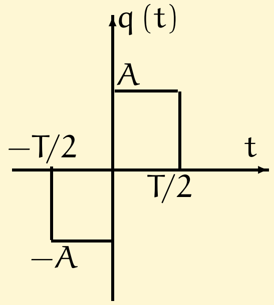
Q2) A second order linear time invariant system is given as

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = \frac{1}{4}x[n-1]$$

Find impulse response of the system ($y[-2] = y[-1] = 0$).

Q3) Find complex Fourier series coefficients of $x(t)$.





$$
\begin{aligned}
\tilde{x}(t) &= \begin{cases} p(t) + jq(t), & -\dfrac{T}{2} \leq t \leq \dfrac{T}{2} \\ 0, & \text{otherwise} \end{cases} \\
x(t) &= \begin{cases} \tilde{x}(t), & -\dfrac{T_o}{2} \leq t \leq \dfrac{T_o}{2} \\ x(t + l \cdot T_o), & l \in Z \end{cases}
\end{aligned}
$$

where $T_o > T$.

Q4) Find the unilateral Laplace transform of

a) $x(t) = e^{-at}, \quad a > 0,$

b) $y(t) = \dfrac{d}{dt} x(t).$

Reminding :

$$\mathcal{Y}(s) = s\,\mathcal{X}(s) - x(0)$$

Q5) Find inverse Laplace transform of,

$$X(s) = \frac{1}{(s+2)^2 (s+3)}, \qquad \text{Re}[s] > -2$$

using partial fraction expansion.

Q6) The magnitude,

$$A(\omega) = \begin{cases} A_o, & -2\pi B \leq \omega \leq 2\pi B \\ 0, & \text{otherwise} \end{cases}$$

and phase,

$$\theta(\omega) = \begin{cases} -\pi/2, & \omega \leq 0 \\ \pi/2, & \omega > 0 \\ 0, & \text{otherwise} \end{cases}$$

of a Fourier transform are given. Find the inverse Fourier transform ($X(\omega) = A(\omega)\, e^{j\theta(\omega)}$).

ANSWERS

A1)

$$-2 \leq k \leq 2$$

$$-1 \leq n - k \leq 3 \quad \rightarrow \quad -1 + k \leq n \leq 3 + k$$

$$-3 \leq n \leq 5$$

$$\begin{aligned} c[n] &= \sum_{k=-2}^{2} a[k]\, b[n-k] \\ &= a[-2]\, b[n+2] + a[-1]\, b[n+1] + a[0]\, b[n] \\ &\quad + a[1]\, b[n-1] + a[2]\, b[n-2] \end{aligned}$$

$$\begin{aligned}
c[-3] &= a[-2]\,b[-1] \\
&= -4 \\
c[-2] &= a[-2]\,b[0] + a[-1]\,b[-1] \\
&= 11 \\
c[-1] &= a[-2]\,b[1] + a[-1]\,b[0] + a[0]\,b[-1] \\
&= -15 \\
c[0] &= a[-2]\,b[2] + a[-1]\,b[1] + a[0]\,b[0] \\
&\quad + a[1]\,b[-1] \\
&= 6 \\
c[1] &= a[-2]\,b[3] + a[-1]\,b[2] + a[0]\,b[1] \\
&\quad + a[1]\,b[0] + a[2]\,b[-1] \\
&= 6 \\
c[2] &= a[-1]\,b[3] + a[0]\,b[2] + a[1]\,b[1] \\
&\quad + a[2]\,b[0] \\
&= -7 \\
c[3] &= a[0]\,b[3] + a[1]\,b[2] + a[2]\,b[1] \\
&= 10 \\
c[4] &= a[1]\,b[3] + a[2]\,b[2] \\
&= -3 \\
c[5] &= a[2]\,b[3] \\
&= 2 \\
c[n] &= 0, \quad n < -3 \text{ and } n > 5
\end{aligned}$$

A2)

$$h[n] - \frac{3}{4}h[n-1] + \frac{1}{8}h[n-2] = \frac{1}{4}\delta[n-1]$$

For $n < 02$,

$$h[n-2] = 2\delta[n-1] - h[n] + 6h[n-1]$$

$$\begin{aligned} n &= -1 \\ h[-3] &= 2\delta[-2] - h[-1] + 6h[-2] \\ &= 0 \end{aligned}$$

$$\begin{aligned} n &= -2 \\ h[-4] &= 2\delta[-3] - h[-2] + 6h[-3] \\ &= 0 \\ &\vdots \\ n &< 0 \\ h[n] &= 0 \end{aligned}$$

For $n \geq 0$,

$$h[n] = \frac{1}{4}\delta[n-1] + \frac{3}{4}h[n-1] - \frac{1}{8}h[n-2]$$

$$\begin{aligned} n &= 0 \\ h[0] &= \frac{1}{4}\delta[-1] + \frac{3}{4}h[-1] - \frac{1}{8}h[-2] \\ &= 0 \end{aligned}$$

$$\begin{aligned} n &= 1 \\ h[1] &= \frac{1}{4}\delta[0] + \frac{3}{4}h[0] - \frac{1}{8}h[-1] \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} n &= 2 \\ h[2] &= \frac{1}{4}\delta[1] + \frac{3}{4}h[1] - \frac{1}{8}h[0] \\ &= \frac{3}{8} \end{aligned}$$

$$\begin{aligned} n &= 3 \\ h[3] &= \frac{1}{4}\delta[2] + \frac{3}{4}h[2] - \frac{1}{8}h[1] \\ &= \frac{3}{4}\cdot\frac{3}{8} - \frac{1}{8}\cdot\frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned}
n &= 4 \\
h[4] &= \frac{1}{4}\delta[3] + \frac{3}{4}h[3] - \frac{1}{8}h[2] \\
&= \frac{3}{4}\cdot\frac{1}{4} - \frac{1}{8}\cdot\frac{3}{8} \\
&= \frac{9}{64}
\end{aligned}$$

$$\begin{aligned}
n &= 5 \\
h[5] &= \frac{1}{4}\delta[4] + \frac{3}{4}h[4] - \frac{1}{8}h[3] \\
&= \frac{3}{4}\cdot\frac{9}{64} - \frac{1}{8}\cdot\frac{1}{4} \\
&= \frac{19}{256} \\
&\vdots \\
n &\geq 0 \\
h[n] &= \frac{2^n - 1}{4^n}
\end{aligned}$$

$$\begin{aligned}
h[n] &= \frac{2^n - 1}{4^n}u[n] \\
&= \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]
\end{aligned}$$

A3)

$$\begin{aligned}
a_k &= \frac{1}{T_o}\int_{-T/2}^{T/2} p(t)\, e^{-j\frac{2\pi}{T_o}k\,t}\,dt + j\,\frac{1}{T_o}\int_{-T/2}^{T/2} q(t)\, e^{-j\frac{2\pi}{T_o}k\,t}\,dt \\
&= \frac{2}{T_o}\int_{0}^{T/2} p(t)\cos\left(\frac{2\pi}{T_o}k\,t\right)dt + \frac{2}{T_o}\int_{0}^{T/2} q(t)\sin\left(\frac{2\pi}{T_o}k\,t\right)dt \\
&= \frac{2A}{T_o}\int_{0}^{T/2} \cos\left(\frac{2\pi}{T_o}k\,t\right)dt + \frac{2A}{T_o}\int_{0}^{T/2} \sin\left(\frac{2\pi}{T_o}k\,t\right)dt
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{2}{T_o}\frac{T_o}{2\pi k}\sin\left(\frac{2\pi}{T_o}k\,t\right)\Bigg|_0^{T/2} - \frac{2}{T_o}\frac{T_o}{2\pi k}\cos\left(\frac{2\pi}{T_o}k\,t\right)\Bigg|_0^{T/2} \\
&= \frac{A}{\pi k}\sin\left(\frac{\pi T}{T_o}k\,t\right) + \frac{A}{\pi k}\left(1-\cos\left(\frac{\pi T}{T_o}k\,t\right)\right)
\end{aligned}$$

A4)

a)

$$\begin{aligned}
\mathcal{X}(s) &= \int_0^{\infty} e^{-a\,t}e^{-s\,t}dt \\
&= -\frac{1}{s+a}e^{-(s+a)\,t}\Bigg|_0^{\infty} \\
&= \frac{1}{s+a}e^{-(s+a)\,\infty} + \frac{1}{s+a} \\
&= \frac{1}{s+a}, \qquad \mathrm{Re}[s+a] > 0, \\
&\qquad\qquad\quad \mathrm{Re}[s] > -a
\end{aligned}$$

b)

$$x(0) = 1.$$

$$\begin{aligned}\mathcal{Y}(s) &= s\cdot\frac{1}{s+a}-1\\ &= -\frac{a}{s+a},\quad \mathrm{Re}[s]>-a\end{aligned}$$

A5)

$$\begin{aligned}X(s) &= \frac{1}{(s+2)^2(s+3)}\\ &= \frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{s+3}\end{aligned}$$

$$\begin{aligned}B &= \lim_{s\to-2}(s+2)^2X(s)\\ &= 1\end{aligned}$$

$$\begin{aligned}C &= \lim_{s\to-3}(s+3)X(s)\\ &= 1\end{aligned}$$

$$\begin{aligned}A &= \lim_{s\to-2}\frac{d}{ds}(s+2)^2X(s)\\ &= \lim_{s\to-2}\frac{d}{ds}\frac{1}{s+3}\\ &= \lim_{s\to-2}-\frac{1}{(s+3)^2}\\ &= -1\end{aligned}$$

$$\begin{aligned}L[t\,y(t)] &= -\frac{d}{ds}Y(s)\\ \frac{1}{(s+2)^2} &= -\frac{d}{ds}\frac{1}{(s+2)}\end{aligned}$$

$$\{\mathrm{Re}\,[s] > -2\} = \{\mathrm{Re}\,[s] > -2\} \cap \{\mathrm{Re}\,[s] > -3\}$$

$$\begin{aligned}
e^{-2t}u(t) &\overset{L}{\longleftrightarrow} \frac{1}{(s+2)}, \quad \mathrm{Re}\,[s] > -2 \\
e^{-3t}u(t) &\overset{L}{\longleftrightarrow} \frac{1}{(s+3)}, \quad \mathrm{Re}\,[s] > -3 \\
t\,e^{-2t}u(t) &\overset{L}{\longleftrightarrow} \frac{1}{(s+2)^2}, \quad \mathrm{Re}\,[s] > -2
\end{aligned}$$

$$x(t) = -e^{-2t}u(t) + t\,e^{-2t}u(t) + e^{-3t}u(t)$$

A6)

$$\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)\,e^{j\theta(\omega)}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-2\pi B}^{0} A_o e^{-j\frac{\pi}{2}}e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{2\pi B} A_o e^{j\frac{\pi}{2}}e^{j\omega t}d\omega \\
&= \frac{A_o}{2\pi}\int_{0}^{2\pi B} e^{-j\frac{\pi}{2}}e^{-j\omega t}d\omega + \frac{A_o}{2\pi}\int_{0}^{2\pi B} e^{j\frac{\pi}{2}}e^{j\omega t}d\omega \\
&= \frac{A_o}{2\pi}\int_{0}^{2\pi B} e^{-j\left(\omega t + \frac{\pi}{2}\right)}d\omega + \frac{A_o}{2\pi}\int_{0}^{2\pi B} e^{j\left(\omega t + \frac{\pi}{2}\right)}d\omega \\
&= \frac{A_o}{\pi}\int_{0}^{2\pi B} \cos\left(\omega t + \frac{\pi}{2}\right)d\omega \\
&= -\frac{A_o}{\pi}\int_{0}^{2\pi B} \sin(\omega t)\,d\omega
\end{aligned}$$

$$\begin{aligned} x(t) &= -\frac{A_o}{\pi}\int_0^{2\pi B}\sin(\omega t)\,d\omega \\ &= \frac{A_o}{\pi}\frac{1}{t}\cos(\omega t)\Big|_0^{2\pi B} \\ &= \frac{A_o}{\pi t}\left(\cos(2\pi B t)-1\right) \end{aligned}$$

Answer all questions. Exam time is 90 minutes.

QUESTIONS

Q1) A first order linear time invariant system is given as

$$y[n] - \frac{1}{4} y[n-1] = x[n] - x[n-1] \quad .$$

Find impulse response of the system ($y[-1] = 0$).

Q2) Two discrete-time signals are given:

n	$a[n]$	$b[n]$
-1	-2	-4
0	3	3
1	-1	3
2	1	-1
elsewhere	0	0

Find convolution, $c[n] = a[n] * b[n]$.

Q3) Find complex Fourier series coefficients of $x(t) = \left(\cos(\pi t) + \sin\left(\frac{1}{2}\pi t\right)\right)^2$.

Q4) Find inverse Laplace transform of,

$$X(s) = \frac{1}{(s+2)(s+3)^2}, \qquad \text{Re}[s] > -2$$

using partial fraction expansion.

Q5) Find the unilateral Laplace transform of $x(t) = \cos(2t)$.

QUESTIONS

Q1) Find trigonometric Fourier series of periodic signal [1],

$$x(t) = 0.5 - [\,|t|\,]_2 = \begin{cases} 0.5 - |t|, & -1 \le t \le 1 \\ x(t - \ell \cdot 2), & \ell \in \mathcal{Z} \end{cases}$$

Q2) Consider that $x(t)$ is an even periodic signal with fundamental period T_o and has zero average value. Then its trigonometric Fourier series representation is

$$x(t) = \sum_{k=1}^{\infty} a_k \cos\left(k \frac{2\pi}{T_o} t\right)$$

Find Fourier series representation for

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

in terms of the series coefficients of $x(t)$.

Q3) Find Fourier transform of a) $\delta(t)$, b) $\delta(t - t_o)$, c) $\sum_{\ell=-\infty}^{\infty} \delta(t - \ell \cdot T_o)$, d) $e^{j\omega_o t}$.

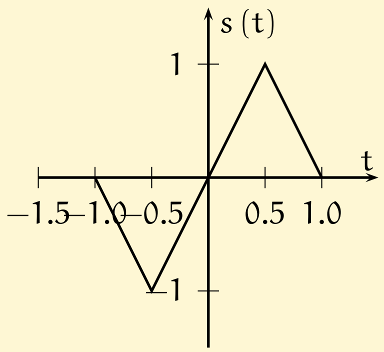
Q4) Find inverse Fourier transform of a) $2\pi\delta(\omega)$, b) $2\pi\delta(\omega - \omega_o)$, c) $\frac{2\pi}{T_o} \sum_{\ell=-\infty}^{\infty} \delta(\omega - \ell \cdot \omega_o)$, d) $e^{-j\omega t_o}$.

Q5) System function of an LTI system is given. The poles of the system function are $s = -5j$ and $s = 5j$. The system function has one zero located at $s = -1$. And $H(0) = \frac{2}{25}$.

[1] The modulo N is denoted by $b = [a]_N$, or in other way $a = b \pmod N$. By definition, their difference is exactly divisible by N ($(a-b)/N$ has no reminder). Note that $b \geq 0$.

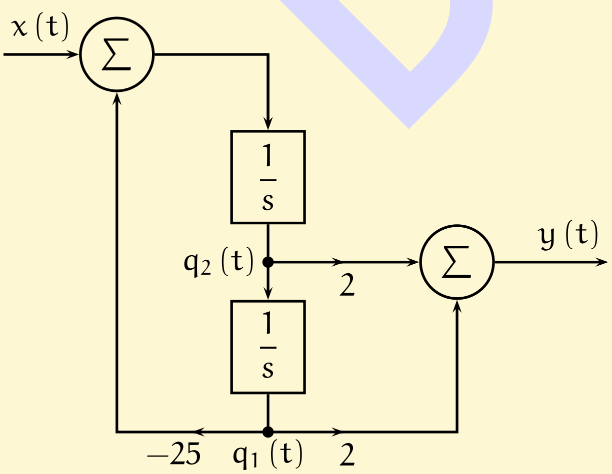
a) Write the system function $H(s)$. b) Write the differential equation describing the system. c) Find the impulse response.

Q6)



A periodic signal $x(t)$ is obtained from the aperiodic signal shown in the following figure; $x(t) = \sum_{\ell=-\infty}^{\infty} s(t - \ell \cdot 2)$. Find period of $y(t) = 2x(t/3) - x(t/5)$.

Q7) A block diagram implementation of an LTI system is given below.



a) Find system function of the system. b) Write state-space equations for the state variables shown in the figure.

Q8) The delta-dirac function is limit case of, $\delta(t) = \lim_{a \to 0} p(t)$.

$$p(t) = \frac{1}{\pi} \frac{a}{a^2 + t^2}, \quad a > 0.$$

The integral and derivative of $p(t)$;

$$\begin{aligned} \int p(t)\, dt &= \int \frac{1}{\pi} \frac{a}{a^2 + t^2}\, dt \\ &= \frac{1}{\pi} \arctan\left(\frac{t}{a}\right). \end{aligned}$$

$$\begin{aligned} d(t) &= \frac{d}{dt} p(t) \\ &= -\frac{2}{\pi} \frac{a\,t}{a^2 + t^2}. \end{aligned}$$

$$\begin{aligned} \lim_{a \to 0} \int_{\infty}^{\infty} f(t)\, \frac{1}{\pi} \frac{a}{a^2 + t^2}\, dt &= \\ \int_{\infty}^{\infty} f(t)\, \delta(t)\, dt &= f(0). \end{aligned}$$

Show that,

$$\begin{aligned} \lim_{a \to 0} \int_{\infty}^{\infty} f(t)\, d(t)\, dt &= \\ \int_{\infty}^{\infty} f(t)\, \delta'(t)\, dt &= -f'(0). \end{aligned}$$

(use partial integration method) and

$$\lim_{a\to 0}\int_{\infty}^{\infty} d\left(t\right)\,dt \;=$$

$$\int_{\infty}^{\infty} \delta'\left(t\right)\,dt \;=\; 0\,.$$

Q9) An LTI system characterized by a differential equation;

$$\frac{d^3}{dt^3}y\left(t\right)+3\frac{d^2}{dt^2}y\left(t\right)+12\frac{d}{dt}y\left(t\right)+10y\left(t\right)=9x\left(t\right)$$

is given.

a) Find the impulse response (by time domain solution of the differential equation). b) Find the system response. c) For state variables, $q_1\left(t\right)=y\left(t\right)$ and $q_2\left(t\right)=\dot{y}\left(t\right)$ and $q_3\left(t\right)=\ddot{y}\left(t\right)$ find state-space equations.

Q10) Trigonometric Fourier series representation of periodic signal

$$x\left(t\right)=\begin{cases} 1, & -1/4\le t\le 1/4 \\ 0, & -1/2\le t<-1/4 \quad \text{and} \quad 1/4<t\le 1/2 \\ x\left(t-\ell\right), & \ell\in\mathcal{Z} \end{cases}$$

is

$$x\left(t\right)=\frac{1}{2}-\sum_{k=1}^{\infty}\frac{2}{\pi\left(2k-1\right)}\left(-1\right)^k\cos\left(2\pi\left(2k-1\right)t\right)\,.$$

Plot the first 2, 5 and, 10 terms of the series for $-1/2\le t\le 1/2$.

Q11) Find trigonometric Fourier series of periodic signal,

$$x(t) = \cos^2(\pi t) \ .$$

Q12) Find Fourier transform of,

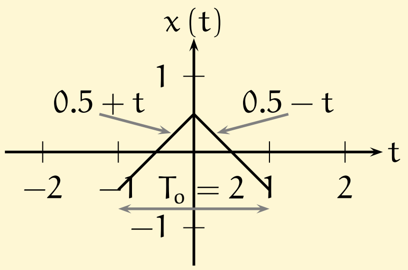
$$x(t) = \begin{cases} 1 - |t|, & -1 \leq t \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

Q13) Find inverse Laplace transform of

$$H(s) = \frac{s+2}{(s+3)(s^2+2s+5)}$$

ANSWERS

A1)



$T_o = 2$ sec and $\omega_o = \dfrac{2\pi}{T_o} = \pi$ rad/sec. The one period of the signal is a triangle shape. It consists of three triangles. The area under one period is

$$\frac{1}{2}\cdot\left(\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\right)\cdot 2+\frac{1}{2}\cdot\left(1\cdot\frac{1}{2}\right)=0\,.$$

Then, $a_0 = 0$. $x(t) = x(-t)$. $x(t)$ is an even function. The trigonometric Fourier series does not contain sin terms. $b_k = 0$.

$$\begin{aligned}
a_k &= \frac{2}{2}\cdot\int_{-1}^{0} x(t)\cos(k\pi t)\,dt + \frac{2}{2}\cdot\int_{0}^{1} x(t)\cos(k\pi t)\,dt \\
&= 2\cdot\int_{0}^{1} x(t)\cos(k\pi t)\,dt, \quad (\,x(t)\cos(k\pi t) \text{ is an even function}) \\
&= 2\cdot\int_{0}^{1}\left(\frac{1}{2}-t\right)\cos(k\pi t)\,dt = \int_{0}^{1}(1-2t)\cos(k\pi t)\,dt
\end{aligned}$$

employ partial integration method, let

$$u = 1-2t, \rightarrow du = -2t\,dt$$
$$dv = \cos(k\pi t)\,dt, \rightarrow v = \frac{1}{\pi k}\sin(k\pi t)$$

$$\begin{aligned} a_k &= (1-2t)\cdot\frac{1}{\pi k}\sin(k\pi t)\Big|_0^1 + 2\cdot\int_0^1 \frac{1}{\pi k}\sin(k\pi t)\,dt \\ &= -\frac{1}{\pi k}\sin(k\pi) - 2\cdot\frac{1}{\pi k}\cdot\frac{1}{\pi k}\cos(k\pi t)\Big|_0^1 \\ &= -\frac{1}{\pi k}\sin(k\pi) - \frac{2}{\pi^2 k^2}\cos(k\pi) + \frac{2}{\pi^2 k^2} \\ &= -\frac{2}{\pi^2 k^2}\cos(k\pi) + \frac{2}{\pi^2 k^2}, && (\sin(k\pi) = 0) \\ &= \frac{4}{\pi^2 k^2}\sin^2\left(k\frac{\pi}{2}\right). && (2\sin^2(\theta) = 1-\cos(2\theta)) \end{aligned}$$

$$a_{2k-1} = \frac{4}{\pi^2(2k-1)^2}, \quad a_{2k} = 0\,.$$

$$x(t) = \sum_{k=1}^{\infty}\frac{4}{\pi^2(2k-1)^2}\cos((2k-1)\pi t)\,.$$

A2)

$$\begin{aligned} y(t) &= \int_{-\infty}^{t} x(\tau)\,d\tau \\ &= \int_{-\infty}^{t}\sum_{k=1}^{\infty} a_k\cos\left(k\frac{2\pi}{T_o}\tau\right)d\tau \\ &= \sum_{k=1}^{\infty} a_k\int_{-\infty}^{t}\cos\left(k\frac{2\pi}{T_o}\tau\right)d\tau \\ &= \sum_{k=1}^{\infty} a_k\frac{T_o}{2\pi k}\sin\left(k\frac{2\pi}{T_o}t\right) \end{aligned}$$

$y(t)$ is an odd function. It has only sin terms and the coefficients, $b_k = a_k\dfrac{T_o}{2\pi k}$.

A3)

a)

$$\begin{aligned}
\mathcal{F}\left[\delta\left(t\right)\right] &= \int_{-\infty}^{\infty} \delta\left(t\right)\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{0^-} \delta\left(t\right)\, e^{-j\omega t}\, dt + \delta\left(0\right)\, e^{-j\omega\cdot 0}\, dt + \int_{0^+}^{\infty} \delta\left(t\right)\, e^{-j\omega t}\, dt \\
&= 0 + \delta\left(0\right)\, dt + 0 \\
&= 1
\end{aligned}$$

b)

$$\begin{aligned}
\mathcal{F}\left[\delta\left(t - t_o\right)\right] &= \int_{-\infty}^{\infty} \delta\left(t - t_o\right)\, e^{-j\omega t}\, dt \\
&= \delta\left(0\right)\, e^{-j\omega\cdot t_o}\, dt \;=\; e^{-j\omega\cdot t_o}
\end{aligned}$$

c)

$$\begin{aligned}
\mathcal{F}\left[\sum_{\ell=-\infty}^{\infty} \delta\left(t - \ell\cdot T_o\right)\right] &= \sum_{\ell=-\infty}^{\infty} \mathcal{F}\left[\delta\left(t - \ell\cdot T_o\right)\right] \\
&= \sum_{\ell=-\infty}^{\infty} e^{-j\omega\cdot\ell\cdot T_o} \\
&= \frac{2\pi}{T_o}\sum_{\ell=-\infty}^{\infty} \delta\left(\omega - \ell\cdot\frac{2\pi}{T_o}\right)
\end{aligned}$$

See the footnote [2].

[2] The Fourier transform sums to zero for $\omega \neq k\cdot\frac{2\pi}{T_o}$ and sums to infinity for $\omega = k\cdot\frac{2\pi}{T_o}$. Using Fourier series;

$$\sum_{\ell=-\infty}^{\infty} e^{j\ell\theta} = \sum_{\ell=-\infty}^{\infty} e^{-j\ell\theta} =$$

$$2\sum_{\ell=0}^{\infty} \cos\left(\ell\theta\right) + 1 = \sum_{\ell=1}^{\infty} 2\pi\delta\left(\theta - \ell\cdot 2\pi\right)$$

d)

$$\begin{aligned}\mathcal{F}\left[e^{j\omega_o t}\right] &= \int_{-\infty}^{\infty} e^{j\omega_o t}\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} e^{-j\,t(\omega-\omega_o)}\, dt \\ &= 2\pi\delta\left(\omega-\omega_o\right)\end{aligned}$$

See the footnote [3].

A4)

a)

$$\mathcal{F}^{-1}\left[2\pi\delta\left(\omega\right)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta\left(\omega\right)\, e^{j\omega t}\, d\omega = 1$$

b)

$$\mathcal{F}^{-1}\left[2\pi\delta\left(\omega-\omega_o\right)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta\left(\omega-\omega_o\right)\, e^{j\omega t}\, d\omega = e^{j\omega_o t}$$

c)

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{2\pi}{T_o}\sum_{\ell=-\infty}^{\infty}\delta\left(\omega-\ell\cdot\frac{2\pi}{T_o}\right)\right] &= \frac{1}{T_o}\sum_{\ell=-\infty}^{\infty}\mathcal{F}^{-1}\left[2\pi\delta\left(\omega-\ell\cdot\frac{2\pi}{T_o}\right)\right] \\ &= \frac{1}{T_o}\sum_{\ell=-\infty}^{\infty} e^{jt\cdot\ell\cdot\frac{2\pi}{T_o}}\end{aligned}$$

[3]The integral sums to zero for $\omega \neq 0$ and sums to infinity for $\omega = 0$. Using Fourier transform;

$$\int_{-\infty}^{\infty} e^{j\,\alpha\,\theta}\, d\alpha = \int_{-\infty}^{\infty} e^{-j\,\alpha\,\theta}\, d\alpha = 2\pi\delta\left(\theta\right)$$

d)

$$\begin{aligned}\mathcal{F}\left[e^{-j\omega t_o}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_o}\, e^{j\omega t}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\,\omega(t-t_o)}\, d\omega \\ &= \delta\left(t-t_o\right)\end{aligned}$$

A5)

a)

$$H(s) = \frac{A\ (s+1)}{(s+5j)\,(s-5j)} = \frac{A\ (s+1)}{s^2+25}$$

$$\begin{aligned}H(0) &= \frac{A}{25} = \frac{2}{25} \\ A &= 2\end{aligned}$$

b)

$$\begin{aligned}\frac{Y(s)}{X(s)} &= \frac{2\ (s+1)}{s^2+25} \\ s^2Y(s) + 25Y(s) &= 2sX(s) + 2X(s)\end{aligned}$$

$$\frac{d^2}{dt^2}y(t) + 25y(t) = 2\frac{d}{dt}x(t) + 2x(t)$$

c)

$$\begin{aligned}
\int_{0^-}^{0^+} \left[\frac{d^2}{dt^2} y(t)\right] dt + 25 \int_{0^-}^{0^+} y(t)\, dt &= 2 \int_{0^-}^{0^+} \left[\frac{d}{dt}\delta(t)\right] dt &&+ 2 \int_{0^-}^{0^+} \delta(t)\, dt \\
y'(0^+) - y'(0^-) + 0 &= 2 \cdot 0 &&+ 2 \\
y'(0^+) &= 2
\end{aligned}$$

See [4]

$$\begin{aligned}
\int_{0^-}^{0^+} \left[\frac{d}{dt} y(t)\right] dt + 25 \int_{0^-}^{0^+} \left[D^{-1} y(t)\right] dt &= 2 \int_{0^-}^{0^+} \delta(t)\, dt &&+ 2 \int_{0^-}^{0^+} u(t)\, dt \\
y(0^+) - y(0^-) + 0 &= 2 &&+ 0 \\
y(0^+) &= 2
\end{aligned}$$

$$\frac{d^2}{dt^2} y(t) + 25 y(t) = 0, \quad \text{for } t > 0$$

The impulse response is the solution of the differential equation for $t > 0$. The characteristic polynomial has complex conjugate roots.

$$y(t) = K^* e^{-j5t} + K e^{j5t}$$

[4]

$$\begin{aligned}
D^{-2} y(t) &= \int_{-\infty}^{t} \int_{-\infty}^{\tau} y(\xi)\, d\xi\, d\tau \\
D^{-1} y(t) &= \int_{-\infty}^{t} y(\tau)\, d\tau \\
D y(t) &= \frac{d}{dt} y(t) \\
D^2 y(t) &= \frac{d^2}{dt^2} y(t)
\end{aligned}$$

$$K = \frac{A}{2} + j\frac{B}{2}$$

$$\begin{aligned} y(t) &= \left(\frac{A}{2} - j\frac{B}{2}\right) e^{-j5t} + \left(\frac{A}{2} + j\frac{B}{2}\right) e^{j5t} \\ &= A\cos(5t) + B\sin(5t) \end{aligned}$$

$$y'(t) = -5A\sin(5t) + 5B\cos(5t)$$

$$\begin{aligned} y(0^+) &= A = 2 \\ y'(0^+) &= 5B = 2 \\ & B = \frac{2}{5} \end{aligned}$$

$$y(t) = 2\cos(5t)\, u(t) + \frac{2}{5}\sin(5t)\, u(t)$$

A6) Fundamental period of $x(t)$ is 2 sec. Consider that T_o is fundemental period of $y(t)$.

$$\begin{aligned} y(t+T_o) &= 2x\left(\frac{t+T_o}{3}\right) - x\left(\frac{t+T_o}{5}\right) \\ &= 2x\left(\frac{t}{3} + \frac{T_o}{3}\right) - x\left(\frac{t}{5} + \frac{T_o}{5}\right) \end{aligned}$$

$T_o/3$ and $T_o/5$ have to be multiple of 2 sec.

$$\begin{aligned} \frac{t}{3} &= \ell \cdot 2 \\ \frac{t}{5} &= k \cdot 2, \qquad \frac{\ell}{k} = \frac{5}{3} \end{aligned}$$

$\ell = 5,\ k = 3 \Rightarrow T_o = 30$ sec.

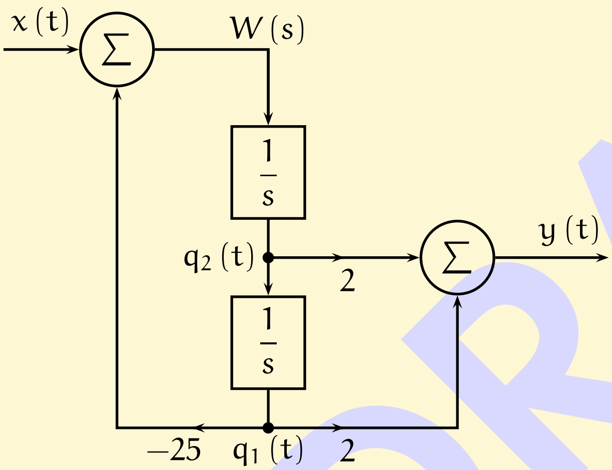
The frequencies of $x(t/3)$ and $x(t/6)$ must be integer multiples (harmonics) of frequency of $y(t)$.

Then,

$$
\begin{aligned}
m \cdot \frac{1}{T_o} &= \frac{1}{6} \\
n \cdot \frac{1}{T_o} &= \frac{1}{10} \qquad \frac{m}{n} = \frac{5}{3}
\end{aligned}
$$

$m = 5,\ n = 3 \Rightarrow \frac{1}{T_o} = \frac{1}{30}$ Hz.

A7)



$$
\begin{aligned}
X(s) - \frac{25}{s^2} W(s) &= W(s) \\
W(s)\left(1 + \frac{25}{s^2}\right) &= X(s) \\
W(s) &= \frac{s^2}{s^2 + 25} X(s)
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{s} W(s) + \frac{2}{s^2} W(s) &= Y(s) \\
\left(\frac{2(s+1)}{s^2 2}\right) W(s) &= Y(s)
\end{aligned}
$$

$$Y(s) = \left(\frac{2(s+1)}{s^2}\right)\frac{s^2}{s^2+25}X(s) \; = \frac{2(s+1)}{s^2+25}X(s)$$

$$H(s) = \frac{2(s+1)}{s^2+25}$$

b)

$$\begin{aligned} \dot{q}_1(t) &= q_2(t) \\ \dot{q}_2(t) &= -25q_1(t) + x(t) \\ y(t) &= 2q_1(t) + 2q_2(t) \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot x(t) \\ y(t) &= \begin{bmatrix} 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \end{aligned}$$

A8)

$$\begin{aligned} \int_{-\infty}^{\infty} \delta(t)\, dt &= \lim_{a\to 0^+} \int_{-\infty}^{\infty} p(t)\, dt \\ &= \lim_{a\to 0^+} \frac{1}{\pi}\arctan\left(\frac{t}{a}\right)\Big|_{-\infty}^{\infty} \\ &= \lim_{a\to 0^+} \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\ &= 1 \end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(t)\,dt &= \lim_{a\to 0^+} \int_{-\infty}^{\infty} d(t)\,dt \\ &= \lim_{a\to 0^+} p(t)\Big|_{-\infty}^{\infty} \\ &= \lim_{a\to 0^+} (p(\infty) - p(-\infty)) \\ &= 0\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt &= \lim_{a\to 0^+} \int_{-\infty}^{\infty} f(t)\,d(t)\,dt \\ &= \lim_{a\to 0^+} \left[f(t)\,p(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt \right] \\ &= \lim_{a\to 0^+} \left[f(\infty)\,p(\infty) - f(-\infty)\,p(-\infty) - f'(0) \right] \\ &= -f'(0)\end{aligned}$$

A9)

a)

$$D^3 y(t) + 3D^2 y(t) + 12Dy(t) + 10y(t) = 9x(t)$$

$$\begin{aligned}\int_{0^-}^{0^+} Dy(t)\,dt + 3\int_{0^-}^{0^+} y(t)\,dt + 12\int_{0^-}^{0^+} D^{-1}y(t)\,dt + 10\int_{0^-}^{0^+} D^{-2}y(t)\,dt &= 9\int_{0^-}^{0^+} D^{-2}\delta(t)\,dt \\ y(0^+) - y(0^-) + 3\cdot 0 + 12\cdot 0 + 10\cdot 0 &= 9\cdot 0 \\ y(0^+) &= 0\end{aligned}$$

See [5]

$$\begin{aligned}
\int_{0^-}^{0^+} D^2y(t)\,dt + 3\int_{0^-}^{0^+} Dy(t)\,dt + 12\int_{0^-}^{0^+} y(t)\,dt + 10\int_{0^-}^{0^+} D^{-1}y(t)\,dt &= 9\int_{0^-}^{0^+} D^{-1}\delta(t)\,dt \\
y'(0^+) - y'(0^-) + 3\cdot\left(y(0^+) - y(0^-)\right) + 12\cdot 0 + 10\cdot 0 &= 9\cdot 0 \\
y'(0^+) &= 0
\end{aligned}$$

$$\begin{aligned}
\int_{0^-}^{0^+} D^3y(t)\,dt + 3\int_{0^-}^{0^+} D^2y(t)\,dt + 12\int_{0^-}^{0^+} Dy(t)\,dt + 10\int_{0^-}^{0^+} y(t)\,dt &= 9\int_{0^-}^{0^+} \delta(t)\,dt \\
y''(0^+) - y''(0^-) + 3\cdot\left(y'(0^+) - y'(0^-)\right) + 12\cdot\left(y(0^+) - y(0^-)\right) + 10\cdot 0 &= 9\cdot 1 \\
y''(0^+) &= 9
\end{aligned}$$

The impulse response is zero input response for,

$$D^3y(t) + 3D^2y(t) + 12Dy(t) + 10y(t) = 0, \quad t \geq 0^+$$
$$y''(0^+) = 9, \quad y'(0^+) = 0, \quad y(0^+) = 0$$

The characteristics polynomial,

$$\lambda^3 + 3\lambda^2 + 12\lambda + 10 = (\lambda + 1)\left(\lambda^2 + 2\lambda + 10\right)$$

The roots are, $\lambda_1 = -1$, $\lambda_2 = -1 + 3j$, and $\lambda_3 = -1 - 3j$.

The estimated solution for real root, $\lambda_1 = a$ is $K\,e^{a\,t}$. The solution for the complex conjugate roots

[5] Integral of a signal is continuos if it has only finite discontinuties. Let $a(t)$ is continuous signal. $b(t) = a(t) + K\cdot u(t)$ is discontinuous at $t = 0$. ($b(0^-) = a(0^-)$, $\quad b(0^+) = a(0^+) + K$).

$c(t) = \int_{-\infty}^{t} b(\tau)\,d\tau = \int_{-\infty}^{t} a(\tau)\,d\tau + K\,t\,u(t)$.

$c(0^-) = c(0^+)$.

$\lambda_2 = a + jb$ and $\lambda_3 = a - jb$ is $A\, e^{a\,t}\, \cos(b\,t) + B\, e^{a\,t} \sin(b\,t)$. Then,

$$y(t) = K\, e^{-t} + A\, e^{-t}\, \cos(3\,t) + B\, e^{-t} \sin(3t)$$

and

$$\begin{aligned} y'(t) &= -Ke^{-t} - Ae^{-t} \cos(3\,t) - 3\,Ae^{-t} \sin(3\,t) - Be^{-t} \sin(3\,t) + 3\,Be^{-t} \cos(3\,t) \\ y''(t) &= Ke^{-t} - 8\,Ae^{-t} \cos(3\,t) + 6\,Ae^{-t} \sin(3\,t) - 8\,Be^{-t} \sin(3\,t) - 6\,Be^{-t} \cos(3\,t) \end{aligned}$$

Employing the initial conditions we get,

$$\begin{aligned} K + A &= 0 \\ -K - A + 3B &= 0 \\ K - 8A - 6B &= 9 \end{aligned}$$

From the first and second equations, $B = 0$ is obtained. And from the first and the third equations it is easly found that $K = 1$, and $A = -1$. The impulse response is then,

$$y(t) = e^{-t}\,(1 - \cos(3\,t))\; u(t)\ .$$

b)

$$\begin{aligned} s^3Y(s) + 3s^2Y(s) + 12sY(s) + 10Y(s) &= 9X(s) \\ H(s) = \frac{Y(s)}{X(s)} &= \frac{9}{s^3 + 3s^2 + 12s + 10} \end{aligned}$$

c) $q_1(t) = y(t)$ and $q_2(t) = \dot{y}(t)$ and $q_3(t) = \ddot{y}(t)$. Then,

$$\begin{aligned}\dot{q}_1(t) &= q_2(t)\\ \dot{q}_2(t) &= q_3(t)\end{aligned}$$

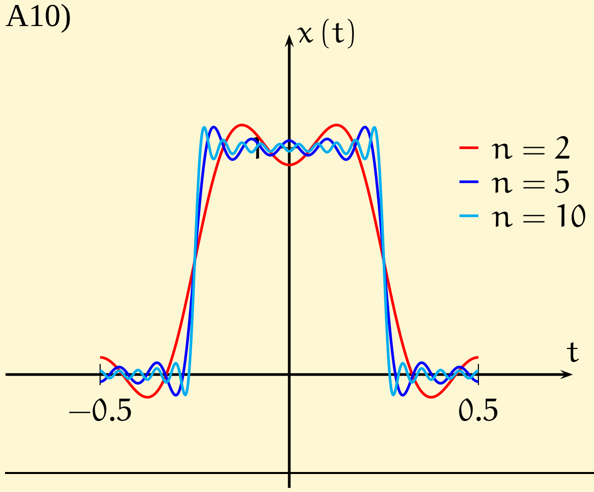
From the differential equation,

$$\begin{aligned}\dot{q}_3(t) + 3q_3(t) + 12q_2(t) + 10q_1(t) &= 9x(t)\\ \dot{q}_3(t) &= -3q_3(t) - 12q_2(t) - 10q_1(t) + 9x(t)\end{aligned}$$

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -12 & -3\end{bmatrix} \cdot \begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \cdot x(t)$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}$$

A10)

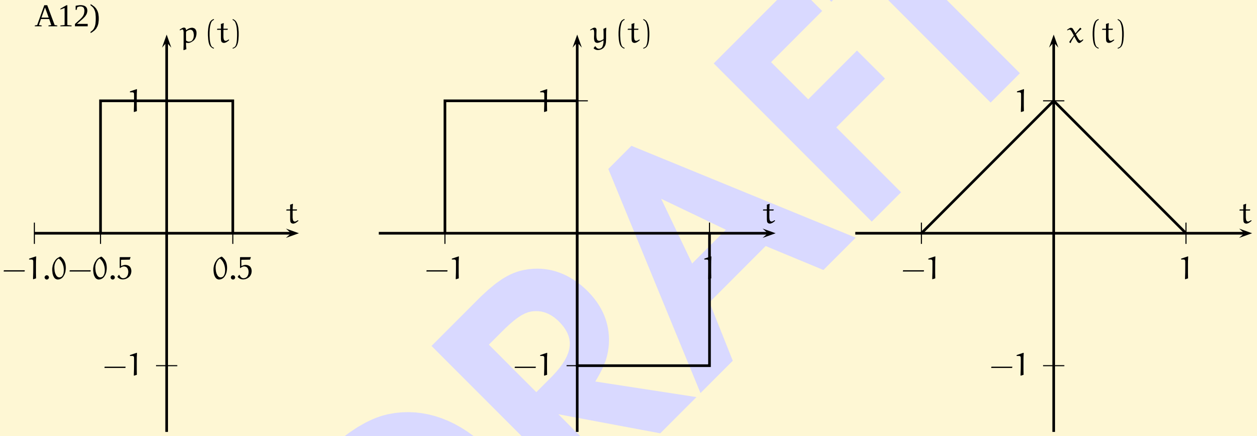


A11)

$$\begin{aligned} x(t) &= \cos^2(\pi t) \\ &= \frac{1}{2} + \frac{1}{2}\cos(2\pi t) \end{aligned}$$

The period of $x(t)$ is $T_o = 1$sec. The trigonometric Fourier series coefficients of are $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}$, and $a_k = 0$ for $k = 2, 3, \ldots, \infty$.

A12)



The signal $x(t)$, can be derived from $p(t)$ and $y(t)$;

$$x(t) = p(t) * p(t)$$

$$x(t) = \int_{-\infty}^{t} y(\tau)\, d\tau = y(t) * u(t)$$

and

$$y(t) = p(t + 0.5) - p(t - 0.5) \ .$$

Then,

$$X(\omega) = P(\omega) \cdot P(\omega)$$

$$P(\omega) = \int_{-1/2}^{1/2} e^{-j\omega t}\, dt$$
$$= 2\int_{0}^{1/2} \cos(\omega t)\ dt = \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right)$$

$$X(\omega) = P^2(\omega)$$
$$= \frac{4}{\omega^2} \sin^2\left(\frac{\omega}{2}\right)$$

A13)

$$H(s) = \frac{s+2}{(s+3)(s^2+2s+5)} = \frac{s+2}{(s+3)(s+1-2j)(s+1+2j)}$$
$$= \frac{A}{s+3} + \frac{B}{s+1-2j} + \frac{C}{s+1+2j}$$
$$= \frac{-1/8}{s+3} + \frac{1/16-(3/16)j}{s+1-2j} + \frac{1/16+(3/16)j}{s+1+2j}$$

$$A = \lim_{s\to-3} (s+3)\ H(s) = -\frac{1}{8}$$
$$B = \lim_{s\to-1+2j} (s+1-2j)\ H(s) = \frac{1}{16} - \frac{3}{16}j$$
$$C = \lim_{s\to-1-2j} (s+1+2j)\ H(s) = \frac{1}{16} + \frac{3}{16}j$$

The inverse Laplace transform;

$$h(t) = -\frac{1}{8}e^{-3t}u(t) + \left(\frac{1}{16} - \frac{3}{16}j\right) e^{(-1+2j)\,t}u(t) + \left(\frac{1}{16} + \frac{3}{16}j\right) e^{(-1-2j)\,t}u(t)$$
$$= -\frac{1}{8}e^{-3t}u(t) + \frac{1}{8}e^{-t}\ (\cos(2t) + 3\sin(2t))\ u(t)\ .$$

QUESTIONS

Q1) Find the Fourier transform of ;

a)

$$x(t) = \begin{cases} 2, & -1 \leq t < 0 \\ -1, & 0 \leq t < 2 \\ 0, & \text{elsewhere} \end{cases}$$

b)

$$y(t) = \begin{cases} 2t + 2, & -1 \leq t < 0 \\ -t + 2, & 0 \leq t < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Q2) Find the trigonometric Fourier series coefficients of;

$$y(t) = \begin{cases} \sin(2\pi t), & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0, & -1 \leq t < -\frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq t < 1 \\ y(t + \ell \cdot 2), & \ell \in \mathcal{Z} \end{cases}$$

Q3) Find inverse Laplace transform of

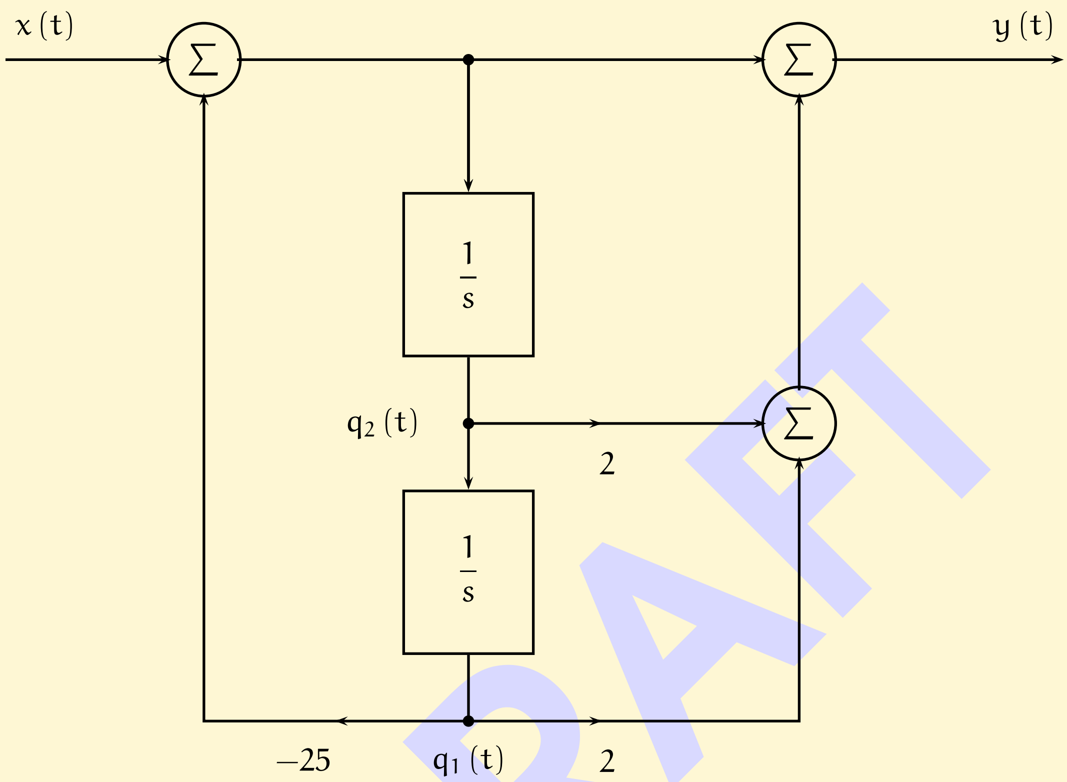
a)

$$H(s) = \frac{1}{(s+2)(s^2+9)}$$

b)

$$H(s) = \frac{s}{(s+2)(s^2+9)}$$

Q4) A block diagram implementation of an LTI system is given below.



a) Find system function of the system. b) Write state-space equations for the state variables shown in the figure.

Q5) An LTI system characterized by a differential equation;

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 10y(t) = 9x(t)$$

is given.

a) Find the impulse response (by time domain solution of the differential equation). b) Find the system response.

ANSWERS

A1)

a)

$$\begin{aligned}
X(\omega) &= \int_{-1}^{0} 2 \cdot e^{-j\omega t}\, dt + \int_{0}^{2} (-1) \cdot e^{-j\omega t}\, dt \\
&= -\frac{2}{j\omega} e^{-j\omega t}\Big|_{-1}^{0} + \frac{1}{j\omega} e^{-j\omega t}\Big|_{0}^{2} \\
&= -\frac{2}{j\omega} + \frac{2}{j\omega} e^{j\omega} + \frac{1}{j\omega} e^{-2j\omega} - \frac{1}{j\omega} \\
&= e^{j\omega/2} \frac{2}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2}\right) + e^{-j\omega} \frac{1}{j\omega} \left(e^{-j\omega} - e^{j\omega}\right) \\
&= e^{j\omega/2} \frac{4}{j\omega} \sin(\omega/2) - e^{-j\omega} \frac{2}{j\omega} \sin(\omega) \\
&= 2\, e^{j\omega/2} \frac{\sin(\omega/2)}{\omega/2} - 2\, e^{-j\omega} \frac{\sin(\omega)}{\omega}
\end{aligned}$$

b)

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$Y(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

$$X(0) = 0$$

$$Y(\omega) = \frac{2}{j\omega} e^{j\omega/2} \frac{\sin(\omega/2)}{\omega/2} - \frac{2}{j\omega} e^{-j\omega} \frac{\sin(\omega)}{\omega}$$

A2) The period is $T_o = 2$ sec and $\omega_o = \pi$ rad/sec.

$$y(t) = -y(-t)$$

The signal is an odd function. The Fourier series does not contain dc and cos terms.

$$\begin{aligned}
b_k &= \int_{-1/2}^{1/2} \sin(2\pi t)\,\sin(k\pi t)\,dt \\
&= -\frac{1}{2}\int_{-1/2}^{1/2} \cos((2+k)\,\pi t)\,dt + \frac{1}{2}\int_{-1/2}^{1/2} \cos((2-k)\,\pi t)\,dt \\
&= -\frac{1}{2}\frac{1}{\pi\,(2+k)}\sin((2+k)\,\pi t)\Big|_{-1/2}^{1/2} + \frac{1}{2}\frac{1}{\pi\,(2-k)}\sin((2-k)\,\pi t)\Big|_{-1/2}^{1/2} \\
&= -\frac{1}{\pi\,(2+k)}\sin\left(\frac{1}{2}\,(2+k)\,\pi\right) + \frac{1}{\pi\,(2-k)}\sin\left(\frac{1}{2}\,(2-k)\,\pi\right) \\
&= \frac{1}{\pi\,(2+k)}\sin\left(\frac{k\pi}{2}\right) + \frac{1}{\pi\,(2-k)}\sin\left(\frac{k\pi}{2}\right) \\
&= \frac{4}{\pi\,(4-k^2)}\sin\left(\frac{k\pi}{2}\right)
\end{aligned}$$

$$b_{2\ell} = 0$$

$$b_{2\ell-1} = \frac{4\,(-1)^{1+\ell}}{\pi\left(4-(2\ell-1)^2\right)}$$

$$y(t) = \sum_{\ell=1}^{\infty} \frac{4\,(-1)^{1+\ell}}{\pi\left(4-(2\ell-1)^2\right)}\sin((2\ell-1)\,\pi t)$$

A3)

a)

$$\frac{1}{(s+2)(s^2+9)} = \frac{A}{s+2} + \frac{B}{s+j3} + \frac{C}{s-j3}$$

$$A = \lim_{s\to-2}(s+2)\,\frac{1}{(s+2)(s^2+9)} = \frac{1}{11}$$

$$B = \lim_{s\to-j3}(s+j3)\,\frac{1}{(s+2)(s^2+9)} = \lim_{s\to-j3}\frac{1}{(s+2)(s-j3)} = \frac{1}{18-j12}$$

$$C = B^* = \frac{1}{18 + j12}$$

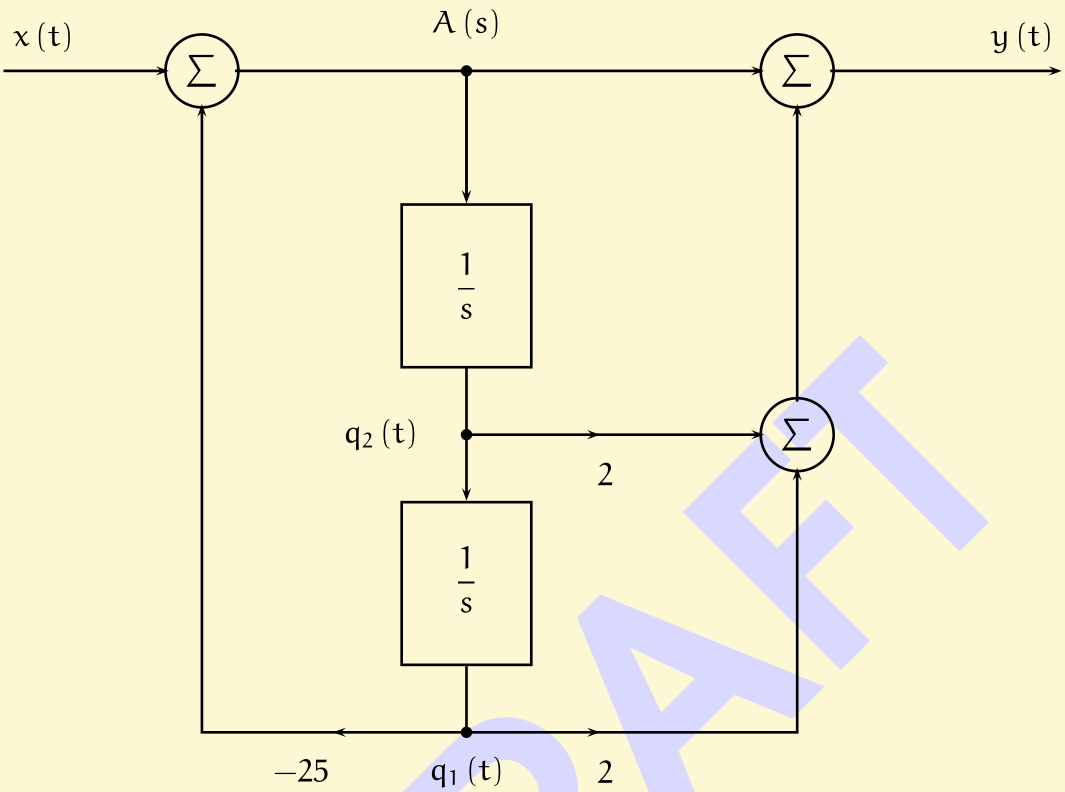
$$\begin{aligned}
h(t) &= \frac{1}{11} e^{-2t} + \frac{1}{18 - j12} e^{-j3t} + \frac{1}{18 + j12} e^{j3t} \\
&= \frac{1}{11} e^{-2t} + \frac{1}{6} \frac{1}{3 - j2} e^{-j3t} + \frac{1}{6} \frac{1}{3 + j2} e^{j3t} \\
&= \frac{1}{11} e^{-2t} + \frac{1}{6 \cdot 11} (3 + j2)\, e^{-j3t} + \frac{1}{6 \cdot 11} (3 - j2)\, e^{j3t} \\
&= \frac{1}{11} e^{-2t} + \frac{1}{11} \left(\cos(3t) + \frac{2}{3} \sin(3t) \right), \qquad t > 0\,.
\end{aligned}$$

b)

$$\mathcal{L}^{-1}\left[\frac{s}{(s+2)(s^2+9)}\right] = \frac{d}{dt}\mathcal{L}^{-1}\left[\frac{1}{(s+2)(s^2+9)}\right]$$

$$h(t) = -\frac{2}{11} e^{-2t} + \frac{1}{11} (-3 \sin(3t) + 2 \cos(3t)), \qquad t > 0\,.$$

A4)



a)

$$A(s) = -\frac{25}{s^2}A(s) + X(s)$$
$$A(s)\left(1 + \frac{25}{s^2}\right) = X(s)$$

$$Y(s) = A(s) + \frac{2}{s}A(s) + \frac{2}{s^2}A(s)$$
$$Y(s) = A(s)\left(1 + \frac{2}{s} + \frac{2}{s^2}\right)$$

$$Y(s) = \frac{s^2+2s+2}{s^2}\,\frac{s^2}{s^2+25}X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 2s + 2}{s^2 + 25}$$

b)

$$\dot{q}_1(t) = q_2(t)$$

$$\dot{q}_2(t) = -25\, q_1(t) + x(t)$$

$$\begin{aligned} y(t) &= -25\, q_1(t) + x(t) + 2\, q_2(t) + 2\, q_1(t) \\ &= -23\, q_1(t) + 2\, q_2(t) + x(t) \end{aligned}$$

A5)

$$D^2 y(t) + 2\, D\, y(t) + 10\, y(t) = 9\, x(t)$$

$$D\, y(t) + 2\, y(t) + 10\, D^{-1} y(t) = 9\, D^{-1} x(t)$$

$$\int_{0^-}^{0^+} [D\, y(t)]\, dt + 2 \int_{0^-}^{0^+} y(t)\, dt + 10 \int_{0^-}^{0^+} \left[D^{-1} y(t)\right] dt = 9 \int_{0^-}^{0^+} \left[D^{-1} \delta(t)\right] dt$$

$$y(0^+) - y(0^-) + 2 \cdot 0 + 10 \cdot 0 = 9 \cdot 0$$

$$y(0^+) = 0$$

$$\int_{0^-}^{0^+} \left[D^2 y(t)\right] dt + 2 \int_{0^-}^{0^+} [D\, y(t)]\, dt + 10 \int_{0^-}^{0^+} y(t)\, dt = 9 \int_{0^-}^{0^+} \delta(t)\, dt$$

$$y'(0^+) - y'(0^-) + 2 \cdot 0 + 10 \cdot 0 = 9 \cdot 1$$

$$y'(0^+) = 9$$

$$\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 10y(t) = 0, \qquad t > 0\,.$$

$$\lambda^2 + 2\lambda + 10 = 0$$

$$\lambda_{1,2} = -1 \pm j3$$

$$y(t) = A\,e^{(-1+j3)\,t} + B\,e^{(-1-j3)\,t}$$

$$y'(t) = A\,(-1+j3)\;e^{(-1+j3)\,t} + B\,(-1-j3)\;e^{(-1-j3)\,t}$$

$$y(0^+) = A + B \;=\; 0$$

$$y'(0^+) = A\,(-1+j3) + B\,(-1-j3) \;=\; 9$$

$$A + B = 0$$

$$-A + B = j3$$

$$A = -\frac{3}{2}j$$

$$B = \frac{3}{2}j$$

$$y(t) = -\frac{3}{2}j\,e^{(-1+j3)\,t} + \frac{3}{2}j\,e^{(-1-j3)\,t}$$

$$= 3\,e^{-t}\,\sin(3\,t)\;, \qquad t > 0\,.$$

QUESTIONS

Q1) Find Fourier transform of the following continuous time signal.

$$y(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & 1 \leq t \leq 2, \\ 0, & \text{otherwise} \end{cases}$$

. . . .

$$\int te^{at}\, dt = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at} + C$$

$$\int e^{at}\, dt = \frac{1}{a}e^{at} + C$$

$$\int_0^1 te^{at}\, dt + \int_1^1 e^{at}\, dt = \frac{1}{a}e^{2a} - \frac{1}{a^2}e^{a} + \frac{1}{a^2}$$

$$\int t\cos(at+b)\, dt = t\frac{1}{a}\sin(at+b) + \frac{1}{a^2}\cos(at+b) + C$$

Q2) Find complex Fourier series of the following periodic signal derived from the signal given in Q1.

$$x(t) = \begin{cases} y(t), & 0 \leq t \leq 2, \\ x(t + \ell \cdot 2), & \ell \in \mathbb{Z} \end{cases}$$

. . . .

$$Y(\omega) = \mathcal{F}[y(t)]$$

$$
\begin{aligned}
a_k &= \frac{1}{T_o} Y\left(\frac{2\pi}{T_o}k\right) \\
&= \frac{1}{2} Y(\pi k)
\end{aligned}
$$

Q3) Find convolution of

$$
e^{-a \cdot t} u(t), \quad a > 0
$$

with itself.

. . . .

$$
\begin{aligned}
z(t) &= y(t) * x(t) \\
&= \int_{-\infty}^{\infty} y(\lambda)\, x(t-\lambda)\ d\lambda
\end{aligned}
$$

$$
\begin{aligned}
z(t) &= \int_{-\infty}^{\infty} e^{-a\lambda} u(\lambda) \cdot e^{-a(t-\lambda)} u(t-\lambda)\ d\lambda \\
&= e^{-at} \int_{-\infty}^{\infty} u(\lambda) \cdot u(t-\lambda)\ d\lambda
\end{aligned}
$$

$$
u(\lambda) \cdot u(t-\lambda) = \begin{cases} 1, & 0 < \lambda < t \\ 0, & \text{otherwise} \end{cases}
$$

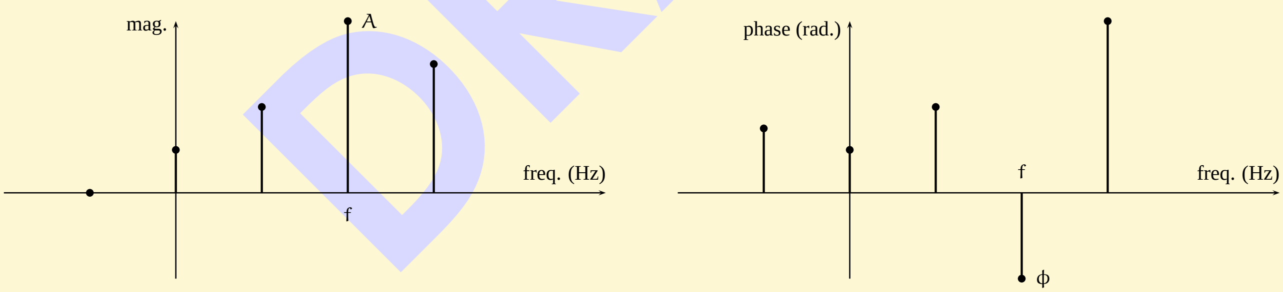
$$\begin{aligned} z(t) &= e^{-at}\int_0^t d\lambda\,, \qquad t>0 \\ &= te^{-at} \end{aligned}$$

$$z(t) = te^{-at}u(t)$$

Q4) Plot one sided discrete magnitude and phase spectrum of the following signal ($A\ \cos(2\pi\cdot f\cdot t+\phi)$ represents an oscillation and A is the magnitude and ϕ is the phase of this oscillation).

$$x(t) = 1 + 2\ \cos(2\pi t + \pi/3) + 3/2\ \cos(4\pi t + 2\pi/3) - 1/2\ \cos(6\pi t)\ .$$

. . . .



Q5) Fourier transform of a continuous time signal is given as

$$X(\omega) = \frac{1-j\omega}{(1+j\omega)(3+j\omega)}\,.$$

Find inverse Fourier transform of the signal by employing partial fraction expension method. Note that

$$\mathcal{F}\left[e^{-a\cdot t}u(t)\right] = \frac{1}{a+j\omega}, \qquad \text{for} \quad a>0\,,$$

and

$$\frac{a+s}{(b+s)\,(c+s)} = \frac{a-b}{c-b}\,\frac{1}{b+s} + \frac{a-c}{b-c}\,\frac{1}{c+s}\,. \quad \checkmark$$

....

The following equation is obtained by multiplying Eq. ($\checkmark$) by minus :

$$\frac{-a-s}{(b+s)\,(c+s)} = \frac{b-a}{c-b}\,\frac{1}{b+s} + \frac{c-a}{b-c}\,\frac{1}{c+s}\,.$$

Comparing this with $X(\omega)$ it is seen that :

$$a = -1, \quad b = 1, \quad c = 3\,.$$

Therefore partial fraction expansion of $X(\omega)$ is

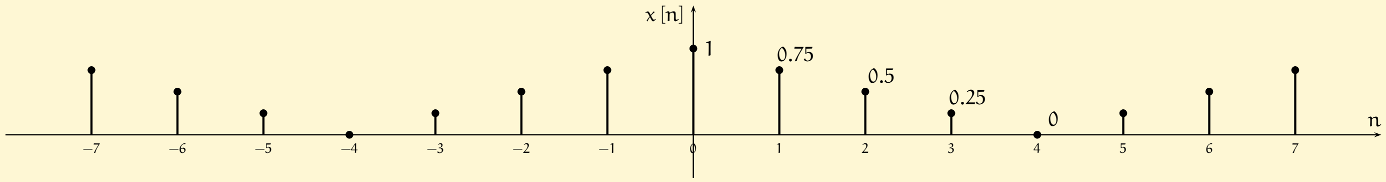
$$\frac{1-j\omega}{(1+j\omega)\,(3+j\omega)} = \frac{1}{1+j\omega} - \frac{2}{3+j\omega}\,.$$

Q6) Generate a discrete time signal by sampling the following continious time signal. The sampling rate is $T_s = 0.5$ sec.

$$x(t) = \begin{cases} 1 - \dfrac{1}{2}\,|t| & -2 \le t \le 2, \\ x(t + \ell \cdot 4)\,, & \ell \in \mathbb{Z} \end{cases}$$

Plot the discrete time signal.

....



$$\begin{aligned} x[n] &= x(n \cdot T_s) \\ &= x(2 \cdot n) \ . \end{aligned}$$

Q7) Test the linearity and causality of the following systems.

a)

$$\frac{d}{dt} y(t) + 2\, y(t) = x(t) \ ,$$

$x(t)$ is input and $y(t)$ is output.

b)

$$y[n-1] + \frac{1}{2} y[n] = x[n] \ .$$

$x[n]$ is input and $y[n]$ is output.

. . . .

A linear system has to satisfy the following property:

Input		Output
x_1	$\longrightarrow$	y_1
x_2	$\longrightarrow$	y_2
$\alpha\, x_1 + \beta\, x_2$	$\longrightarrow$	$\alpha\, y_1 + \beta\, y_2$

If initials values of an differentiation equation (difference equation in discrete time case) are zero the homogenous solution of the differential equation is zero. This means the response is zero for $t < 0$. So the system described by the differential equation is causal. In other case when the differential equation has non-zero initial values causality will be violated. The linearity will also be failed. Because the differential equation response will be composed of particular solution (a linear time invariant and casual system response to the input $x(t)$) and homogenous solution.

Q8) Can the following Fourier series representation of a periodic signal converge ? Why ?

$$x(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \cos(2\pi n t) \ .$$

....

$$\begin{aligned} \sum_{n=11}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1} \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=2}^{\infty} \frac{1}{n} \\ &= 1 \ . \end{aligned}$$

Q9) The unit step function can be considered as limit of $e^{-a \cdot t} u(t)$:

$$u(t) = \lim_{a \to 0^+} e^{-a \cdot t} u(t) \ .$$

a) Using this property find the Fourier transform ($U(\omega)$) of $u(t)$. What is

$$\lim_{\omega \to 0} U(\omega) = \quad ?$$

b) The following relation

$$u(t) = 1 - u(-t) \ ,$$

yields

$$U(\omega) + U(-\omega) = 2\pi\delta(\omega) \ .$$

The above result shows that the Fourier transform of unit step function should be

$$U(\omega) = A(\omega) + \frac{1}{j\omega} .$$

Find $A(\omega)$.

Q10) Find

a) the Fourier transform of $\delta(t)$ and $e^{j(2\pi/T_o)\,k\cdot t}$,

b) and the inverse Fourier transform of $\delta(\omega)$ and $\delta(\omega - (2\pi/T_o)\,k)$.

Q11) Does Fourier transform of

$$x(t) = \frac{1}{1+t} u(t)$$

exist ? Find,

a)

$$\int_{-\infty}^{\infty} |x(t)| \, dt$$

b)

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Q12) Does Fourier transform of

$$x[n] = \frac{1}{1+n} u[n]$$

exist ? Find,

a)

$$\sum_{-\infty}^{\infty} |x[n]|$$

b)

$$\sum_{-\infty}^{\infty} |x[n]|^2$$

Q13) Find impulse response of the following system.

$$y^{''}(t) + 2y^{'}(t) + 5y(t) = x(t) \ .$$

Q14) Find complex Fourier series of the following periodic signal.

$$x(t) = \begin{cases} -\cos(\pi t), & 0 \leq t \leq 1, \\ x(t+\ell), & \ell \in \mathbb{Z}. \end{cases}$$

QUESTIONS

Q1) Frequency response of a linear time invariant and casual continuous-time system is given as follows:

$$H(\omega) = \frac{1}{(j\omega)^2 + 2j\omega + 5}.$$

a) Find the corresponding differential equation of the frequency response. (10p)

b) Find the impulse response of the system by calculating inverse Fourier transform of the frequency response. (15p)

. . . .

a)

$$\begin{aligned} H(\omega) &= \frac{Y(\omega)}{X(\omega)} \\ &= \frac{1}{(j\omega)^2 + 2j\omega + 5} \end{aligned}$$

$$\left[(j\omega)^2 + 2j\omega + 5\right] Y(\omega) = X(\omega)$$
$$(j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + 5Y(\omega) = X(\omega)$$

$$y''(t) + 2y'(t) + 5y(t) = x(t) .$$

b) $H(\omega)$ is partitioned as follows.

$$\begin{aligned} H(\omega) &= \frac{1}{(j\omega)^2 + 2j\omega + 5} \\ &= \frac{1}{(j\omega + 1 + 2j)\,(j\omega + 1 - 2j)} \\ &= \frac{A}{j\omega + 1 + 2j} + \frac{B}{j\omega + 1 - 2j} \end{aligned}$$

$$\begin{aligned} A &= \lim_{j\omega \to -1-2j} (j\omega + 1 + 2j) \cdot H(\omega) \\ &= \lim_{j\omega \to -1-2j} \frac{1}{j\omega + 1 - 2j} = -\frac{1}{4j} \end{aligned}$$

$$\begin{aligned} B &= \lim_{j\omega \to -1+2j} (j\omega + 1 - 2j) \cdot H(\omega) \\ &= \lim_{j\omega \to -1+2j} \frac{1}{j\omega + 1 + 2j} = \frac{1}{4j} \end{aligned}$$

$$H(\omega) = \frac{0.25j}{j\omega + 1 + 2j} - \frac{0.25j}{j\omega + 1 - 2j}$$

$$\begin{aligned} h(t) &= 0.25j\, e^{(-1-2j)t}\, u(t) - 0.25j\, e^{(-1+2j)t}\, u(t) \\ &= e^{-t} \left[0.25j\, e^{-2jt} - 0.25j\, e^{2jt}\right] u(t) \\ &= \frac{1}{2} e^{-t} \sin(t)\; u(t)\ . \end{aligned}$$

Q2) Complex Fourier series coefficients of a periodic signal with period of 2 sec. are given as follows:

$$a_0 = -1, \qquad a_1 = 1 \qquad a_{-1} = 1$$
$$a_2 = -0.5j \qquad a_{-2} = 0.5j\,.$$

Find the periodic signal. (20p)

. . . .

$$\begin{aligned} x(t) &= a_{-2}e^{\frac{2\pi}{T_0}\cdot(-2)\,t} + a_{-1}e^{\frac{2\pi}{T_0}\cdot(-1)\,t} + a_0 + a_1 e^{\frac{2\pi}{T_0}\,t} + a_2 e^{\frac{2\pi}{T_0}\cdot 2\cdot t} \\ &= 0.5j\cdot e^{-2\pi t} + 1\cdot e^{-\pi t} - 1 + 1\cdot e^{\pi t} - 0.5j\cdot e^{2\pi t} \\ &= -1 + 2\cos(\pi t) + \sin(2\pi t)\ . \end{aligned}$$

Q3) Step response of a linear time invariant system is given as follows:

$$s(t) = e^t\ \sin(t)\cdot u(t) - \frac{1}{2}\cos(t)\ .$$

a) Is the system stable ? Why ?

b) Is the system causal ? Why ?

. . . .

a) $\lim\limits_{t\to\infty} s(t) \longrightarrow \infty$. System is unstable. (7.5p)

b) Input is zero for $t < 0$ but output $s(t) = -\frac{1}{2}\cos(t) \neq 0 \quad \text{for} \quad t < 0$. System is non-causal. (7.5p)

Q4) Find complex Fourier series coefficients (not the trigonometric Fourier series coefficients) of the following periodic signal: (20p)

$$x(t) = \begin{cases} 1, & 0 < t < \frac{1}{2} \\ -1, & -\frac{1}{2} < t < 0 \\ x(t+\ell), & \ell \in \mathcal{Z}\,. \end{cases}$$

. . . .

$$\begin{aligned} a_k &= -\int_{-1/2}^{0} e^{-j2\pi kt}\,dt + \int_{0}^{1/2} e^{-j2\pi kt}\,dt \\ &= -\int_{0}^{1/2} e^{j2\pi kt}\,dt + \int_{0}^{1/2} e^{-j2\pi kt}\,dt \\ &= \int_{0}^{1/2} \left(-e^{j2\pi kt} + e^{-j2\pi kt}\right)\,dt \end{aligned}$$

$$\begin{aligned} a_k &= -2j\int_{0}^{1/2} \sin(2\pi kt)\,dt \\ &= \frac{2j}{2\pi k}\cos(2\pi kt)\Big|_{0}^{1/2} \\ &= \frac{1}{j\pi k}\left(1-\cos(\pi k)\right) \end{aligned}$$

$$\begin{aligned} a_{2k-1} &= \frac{2}{j\pi(2k-1)} \\ a_{2k} &= 0, \qquad k \in \mathcal{Z}\,. \end{aligned}$$

Q5) A linear time invariant system is described by the following constant coefficient linear differential equation.

$$y'(t) + 2y(t) = x(t) \ .$$

a) Find impulse response of the system by solving the differential equation in time domain. (10p)

b) Find the response for the input (15p)

$$x(t) = e^{-t} u(t) \ .$$

. . . .

a)

$$\int_{0^-}^{0^+} y'(t)\, dt + 2\int_{0^-}^{0^+} y(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt$$

$$y(t)\Big|_{0^-}^{0^+} + 2 \cdot 0 = u(t)\Big|_{0^-}^{0^+}$$

$$y(0^+) - y(0^-) = u(0^+) - u(0^-)$$

$u(0^-) = 0$ and, $y(0^-) = 0$ since the system is causal. Therefore,

$$y(0^+) = 1 \ .$$

The characteristic equation of the differential equation is (the n th derivative of $y(t)$: $y^{(n)}(t)$ is replaced by D^n):

$$D + 2 = 0$$

The root of the characteristic equation is $D = -2$. The homogenous solution of the system is then

$$y(t) = Ae^{-2t}, \qquad t > 0\,.$$

From the initial condition unknown constant A is found:

$$y(0^+) = A \ = 1\,.$$

The impulse response of the system is

$$y(t) = e^{-2t}u(t)\ .$$

b)

$$\begin{aligned} y(t) &= e^{-2t}u(t) * e^{-t}u(t) \\ &= \int_0^t e^{-2\lambda} \cdot e^{-(t-\lambda)}\, d\lambda \\ &= e^{-t} \int_0^t e^{-\lambda}\, d\lambda \\ &= -e^{-t} \cdot e^{-\lambda}\Big|_0^t \\ &= -e^{-t} \cdot \left(e^{-t} - 1\right) \\ &= e^{-t} - e^{-2t}, \qquad t > 0\,. \end{aligned}$$

$$y(t) = \left(e^{-t} - e^{-2t}\right) u(t)\ .$$

Q6) Find the Fourier transform of

$$e^{-(a+jb)t}u(t)\ .$$

Does the signal have Fourier transform for all $a \in \mathcal{R}$. If not, what is the interval of $a \in \mathcal{R}$ where Fourier transform exist ? Where complex number $-a-jb$ falls in the comlex plane for this interval of a.

Q7)

a) Does inverse Fourier transform of $\delta(\omega-\omega_o)$ exist ? If yes, find the inverse Fourier transform.

b) Using the result above find Fourier transform of

$$\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt}\ .$$

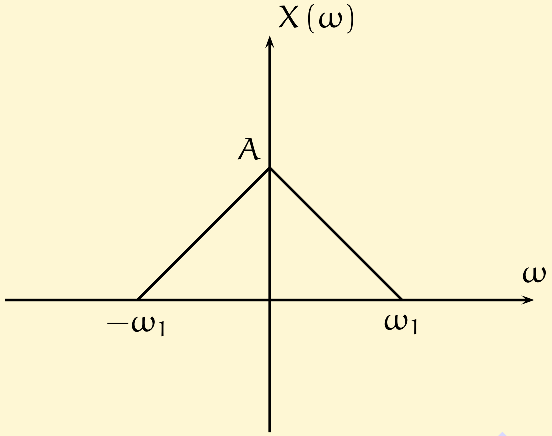
Q8) Find Fourier transform of

$$y(t) = x(t)\cos(\omega_o t)$$

in terms of $X(\omega) = \mathcal{F}[x(t)]$. For

$$X(\omega) = \begin{cases} A + \dfrac{A}{\omega_1}t, & -\omega_1 < t < 0 \\ A - \dfrac{A}{\omega_1}t, & 0 < t < \omega_1 \\ 0, & \text{otherwise} \end{cases}$$

plot $Y(\omega)$.



Q9) Find complex Fourier series of the following periodic impulse train.

$$x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t - \ell \cdot T_o), \qquad \ell \in \mathcal{Z}.$$

Q10) Find step response of the following system.

$$y'(t) + a\,y(t) = y(t)\ .$$

You don't need to calculate $y(0^+)$. Only an impulse can impose initial values at $t = 0^+$.

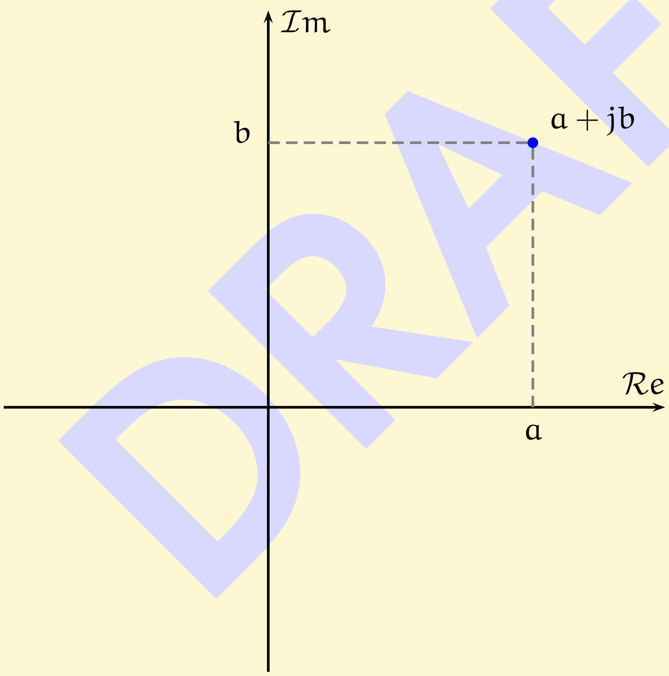
$$\int_{-\infty}^{\infty} f(t)\,\delta(t - t_o)\ dt = f(t_o)$$

$$\mathcal{F}\left[e^{-(a+jb)t}u(t)\right] = \frac{1}{j\omega + a + jb}, \qquad a > 0$$

$$\mathcal{F}[x'(t)] = j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1, \qquad \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega = 2\pi\delta(t)$$

$$\begin{aligned}\cos(\alpha) &= \frac{1}{2}e^{-j\alpha} + \frac{1}{2}e^{j\alpha}\\ \sin(\alpha) &= -\frac{1}{2j}e^{-j\alpha} + \frac{1}{2j}e^{j\alpha}\end{aligned}$$

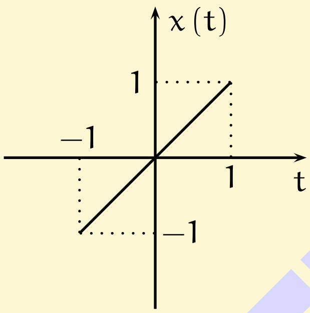


Complex plane

QUESTIONS

Q1)

a) Find Fourier transform of the above signal (15 p) .



b) A periodic signal $y(t)$ is generated from aperiodic signal $x(t)$;

$$y(t) = \sum_{\ell=-\infty}^{\infty} x(t - \ell \cdot 2) \ .$$

Find the complex Fourier series coefficients of this signal by employing the Fourier transform obtained in Q1a (5 p) .

Supplementary:

$$\int te^{at}\, dt = \frac{1}{a} te^{at} - \frac{1}{a^2} e^{at} + c$$

Q2)

a) Find inverse Fourier transform of the following Fourier transform by using partial fraction expansion method (10 p) .

$$Y(\omega) = \frac{2}{1 + \omega^2}$$

Hint: $Y(\omega) \in \mathcal{R}$. Therefore $Y(\omega) = X(\omega) + X^*(\omega)$. For real signals, $\mathcal{F}[x(-t)] = X^*(\omega)$.

b) Find the Fourier transform of

$$z(t) = \frac{2}{1 + t^2}$$

by using duality property of the Fourier transform (10 p) .

Supplementary:

The duality property:

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega)$$
$$z(t) = Y(t) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega) = 2\pi y(-\omega)$$

$$e^{at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{-a + j\omega}, \quad a < 0\,.$$

Q3) Find impulse response of the following causal LTI systems. Integrating both side twice and once in the interval of $[0^-,\ 0^+]$ yields initial conditions; $y'(0^+) = 1$ and $y(0^+) = 0$. $a > 0$ and $b > 0$. Solve the problem in time domain.

a) $y''(t) + (a+b)\,y'(t) + aby(t) = x(t)$ (10 p) .

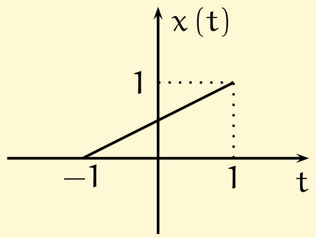
b) $y''(t) + 2ay'(t) + \left(a^2 + b^2\right) y(t) = x(t)$ (10 p) .

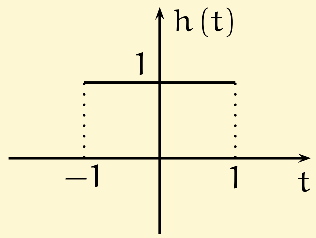
Supplementary:

Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\,\sin(\theta)$$

Q4) Find convolution of the above signals (20 p) .





Hint: Evaluate the convolution integral for the intervals; I. $t < -2$, II. $-2 \le t < 0$, III. $0 \le t < 2$, IV. $t \ge 2$.

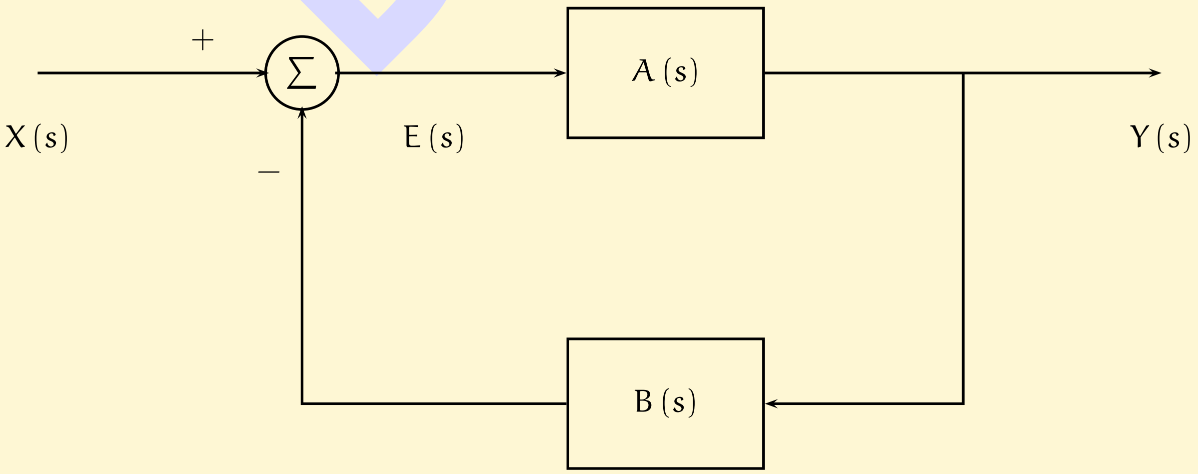
Q5) Transfer function of a causal LTI system is given as following. a) Find Poles, zeros and ROC of $H(s)$. Show poles, zeros and ROC in the s-plane (10 p) . b) Find the impulse response of the system by using partial fraction expansion method (10 p) .

$$H(s) = \frac{s+2}{(s+3)(s+4)^2}$$

Supplementary:

$$\mathcal{L}\left[e^{at}u(t)\right] = \frac{1}{-a+s}, \quad \text{ROC} = \text{Re}[s] > a$$
$$\mathcal{L}\left[te^{at}u(t)\right] = \frac{1}{(-a+s)^2}, \quad \text{ROC} = \text{Re}[s] > a$$

Q6) The following causal LTI system is given.



$$\begin{aligned} A(s) &= \frac{1}{a} \\ B(s) &= \frac{s^2}{a} \end{aligned}$$

For $a = 1$,

a) find the transfer function of the system.

b) find the impulse response of the system.

c) Check the stability. Is system stable ?

Q7) A causal LTI system is given as following

$$2y''(t) + 10y'(t) + 12y(t) = 2x'(t) + 3x(t)$$

a) Find the transfer function of the system.

b) Use partial fraction expansion method to find the inverse Laplace transform of the transfer function and consequently impulse response of the system.

Q8) A causal LTI system is given as following

$$a_0 y''(t) + a_1 y'(t) + a_2 y(t) = bx(t)$$

New variables which are called as state variable can be generated as below

$$\begin{aligned} z_1(t) &= y(t) \\ z_2(t) &= y'(t) = z_1'(t) \end{aligned}$$

The derivative of $z_2(t)$ can be extracted from the differential equation

$$a_0 z_2'(t) + a_1 z_2(t) + a_2 z_1(t) = bx(t)$$
$$z_2'(t) = -\frac{a_1}{a_0} z_2(t) - \frac{a_2}{a_0} z_1(t) + \frac{b}{a_0} x(t)$$

These equations leads to the following matrix equation

$$\begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{bmatrix} \cdot \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{b}{a_0} \end{bmatrix} x(t)$$

The modest form is

$$\underline{z}'(t) = A \cdot \underline{z}(t) + \underline{b} x(t)$$

This equation is known as state equation and matrix A is called as state-transition matrix. The solution of this vector differential equation is

$$\underline{z}(t) = e^{A(t-t_o)} \underline{z}(t_o) + e^{At} \int_{t_o}^{t} e^{-A\lambda} \underline{b} x(\lambda)\, d\lambda$$

Here, e^{At} is the matrix exponential. The matrix exponential is a matrix function on square matrices:

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n}{n!} t^n$$

Find the state variables for the constants, $a_0 = 1$, $a_1 = 3$, $a_2 = 2$, and $b = 1$.

Q8)The following signal

$$x(t) = \cos(2\pi t) + \cos(4\pi t) + \cos(6\pi t)$$

is input to a filter with a frequency response of $H(f)$. The values of the frequency response at $f = 1$, $f = 2$, and $f = 3$ Hz are

$$\begin{aligned} H(1) &= 1 \cdot e^{-j\pi/4} \\ H(2) &= \frac{1}{3} \cdot e^{-j\pi/2} \\ H(3) &= \frac{1}{5} \cdot e^{-j3\pi/4} \end{aligned}$$

a) Find the Fourier transform of $x(t)$.

b) Find the output $y(t)$ of the filter by using the convolution property of the Fourier transform.

ANSWERS

A1)

a)

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \\
&= \int_{-1}^{1} t e^{-j\omega t} dt \\
&= \left. \frac{t}{-j\omega} e^{-j\omega t} \right|_{-1}^{1} - \left. \frac{1}{(-j\omega)^2} e^{-j\omega t} \right|_{-1}^{1} \\
&= -\frac{1}{j\omega} e^{-j\omega} - \frac{1}{j\omega} e^{j\omega} + \frac{1}{\omega^2} e^{-j\omega} - \frac{1}{\omega^2} e^{j\omega} \\
&= \frac{2j}{\omega} \cos(\omega) - \frac{2j}{\omega^2} \sin(\omega) \\
&= \frac{2j}{\omega^2} \left(\omega \cos(\omega) - \sin(\omega)\right)
\end{aligned}
$$

b) Since the periodic signal is generated from the aperiodic signal the complex Fourier series coefficients can be obtained from the Fourier transform of the aperiodic signal.

$$
\begin{aligned}
a_k &= \frac{1}{T_o} X\left(\frac{2\pi}{T_o} k\right) \\
T_o &= 2 \\
\frac{2\pi}{T_o} k &= \pi k
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{2} X(\pi k) \\
&= -\frac{1}{j\pi k} \cos(\pi k) + \frac{1}{j\pi^2 k^2} \sin(\pi k) \\
&= j \frac{\pi k \cos(\pi k) - \sin(\pi k)}{\pi^2 k^2}
\end{aligned}
$$

The above equation gives coefficients for all $k \in \mathcal{Z}$. It can be further simplified by replacing $\cos(\pi k) = (-1)^k$ and $\sin(\pi k) = 0$. Notice that $a_k = \frac{0}{0}$, when $k = 0$, therefore undefined. Actually as $y(t)$ is an odd signal we know that $a_0 = 0$. Applying L'Hospital's rule we can find this. But be careful, a_k is a discrete function of integer k, not a a function of a continuous variable. Hence simplifications is valid for discrete k but limit is defined for continuous variables and we may not obtain true value of a_0 after simplifications. A a result, we separate a_0 from the simplified version of a_k.

$$
a_k = \begin{cases} -\frac{1}{j\pi k}(-1)^k, & k \in \mathcal{Z} - \{0\} \\ 0, & k = 0 \end{cases}
$$

A2)

a) The $Y(\omega)$ can be partitioned into partial fractions as below.

$$
\begin{aligned}
Y(\omega) &= \frac{2}{1+\omega^2} \\
&= \frac{A}{1-j\omega} + \frac{B}{1+j\omega}
\end{aligned}
$$

$$
\begin{aligned}
A &= \lim_{j\omega \to 1} (1-j\omega) \frac{2}{1+\omega^2} = \lim_{j\omega \to 1} \frac{2}{1+j\omega} = 1 \\
B &= \lim_{j\omega \to -1} (1+j\omega) \frac{2}{1+\omega^2} = \lim_{j\omega \to -1} \frac{2}{1-j\omega} = 1
\end{aligned}
$$

$$
Y(\omega) = \frac{1}{1-j\omega} + \frac{1}{1+j\omega}
$$

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{1}{1+j\omega}\right] &= e^{-t}u(t)\\ \mathcal{F}^{-1}\left[\frac{1}{1-j\omega}\right] &= \mathcal{F}^{-1}\left[\left(\frac{1}{1+j\omega}\right)^{*}\right] = \left[e^{-t}u(t)\right]_{t\to -t}\\ &= e^{t}u(-t)\end{aligned}$$

Therefore,

$$\begin{aligned}y(t) &= e^{-t}u(t)+e^{t}u(-t)\\ &= e^{-|t|}\end{aligned}$$

b) Here,

$$z(t)=\left[Y(\omega)\right]_{\omega\to t} = \frac{2}{1+t^2}$$

We can utilize the duality property to find the Fourier transform of $z(t)$.

$$Z(\omega)=2\pi\left[y(t)\right]_{t\to -\omega} = 2\pi e^{-|\omega|}$$

A3) The impulse response, as it appears in its name, is response to the impulse at the input. Impulse is exerted at time $t=0$ and disappears after ($t>0$). Since the system is causal the output (response) is zero for $t<0$. The response for $t>0$ need to be computed and can be found by solving the homogenous differential equation (response to zero input).

a)

$$y''(t)+(a+b)y'+aby(t)=0, \quad t>0$$

The characteristic equation is

$$D^2+(a+b)D+ab=0$$

The roots of the characteristic equation

$$\begin{aligned} (D+a)(D+b) &= 0 \\ D_1 &= -a \\ D_2 &= -b \end{aligned}$$

Then the solution is

$$y(t) = C_1 e^{D_1 t} + C_2 e^{D_2 t} = C_1 e^{-at} + C_2 e^{-bt}$$

The unknown coefficients are found from the initial values imposed by the impulse at $t = 0^+$.

$$\begin{aligned} y(t) &= C_1 e^{-at} + C_2 e^{-bt} \\ y'(t) &= -aC_1 e^{-at} - bC_2 e^{-bt} \end{aligned}$$

$$\begin{aligned} y(0^+) &= C_1 + C_2 = 0 \\ y'(0^+) &= -aC_1 - bC_2 = 1 \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{1}{-a+b} \\ C_2 &= \frac{1}{a-b} \end{aligned}$$

Finally we get,

$$y(t) = \frac{1}{-a+b} e^{-at} u(t) + \frac{1}{a-b} e^{-bt} u(t)$$

b) The same steps are repeated to find the impulse response.

$$y''(t) + 2ay'' + \left(a^2 + b^2\right) y(t) = 0, \quad t > 0$$

The characteristic equation is

$$D^2 + 2aD + \left(a^2 + b^2\right) = 0$$

The roots of the characteristic equation

$$\begin{aligned} (D + a + jb)(D + a - jb) &= 0 \\ D_1 &= -a - jb \\ D_2 &= -a + jb \end{aligned}$$

Then the solution is

$$y(t) = C_1 e^{D_1 t} + C_2 e^{D_2 t} = C_1 e^{(-a-jb)t} + C_2 e^{(-a+jb)t}$$

The unknown coefficients are found from the initial values imposed by the impulse at $t = 0^+$.

$$\begin{aligned} y(t) &= C_1 e^{(-a-jb)t} + C_2 e^{(-a+jb)t} \\ y'(t) &= (-a - jb) C_1 e^{(-a-jb)t} + (-a + jb) C_2 e^{(-a+jb)t} \end{aligned}$$

$$\begin{aligned} y(0^+) &= C_1 + C_2 = 0 \\ y'(0^+) &= (-a - jb) C_1 + (-a + jb) C_2 = 1 \end{aligned}$$

$$\begin{aligned} C_1 &= -\frac{1}{2jb} \\ C_2 &= \frac{1}{2jb} \end{aligned}$$

Replace the coefficients in $y(t)$

$$\begin{aligned} y(t) &= -\frac{1}{2jb}e^{(-a-jb)t} + \frac{1}{2jb}e^{(-a+jb)t} \\ y(t) &= \frac{1}{b}e^{-at}\left[-\frac{1}{2j}e^{-jbt} + \frac{1}{2j}e^{jbt}\right] \\ &= \frac{1}{b}e^{-at}\sin(bt) \end{aligned}$$

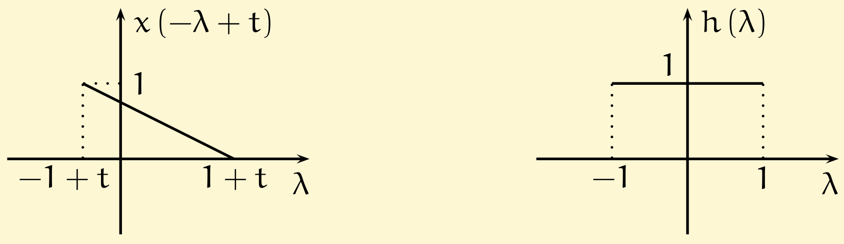
We obtain

$$y(t) = \frac{1}{b}e^{-at}\sin(bt)\,u(t)$$

A4) The convolution integral:

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\,x(t-\lambda)\,d\lambda$$

The graphical illustration of $x(t-\lambda)$ and $h(\lambda)$ are as shown below. As both signals has finite



supports (exist only in finite interval) the integral should be calculated for specified time intervals.

I. For $t < -2$, $h(\lambda) \cdot x(t-\lambda) = 0$. Hence, $y(t) = 0$.

II. For $-2 \leq t < 0$,

$$y(t) = \int_{-1}^{1+t} h(\lambda)\, x(t-\lambda)\, d\lambda = \frac{1}{2} \cdot \frac{1}{2}(2+t) \cdot (2+t) = \frac{1}{4}(2+t)^2$$

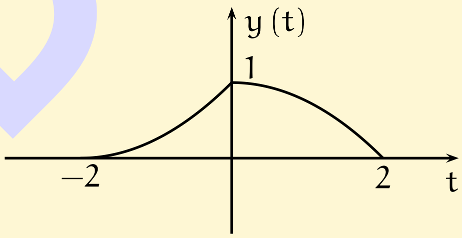
III. For $0 \leq t < 2$,

$$y(t) = \int_{-1+t}^{1} h(\lambda)\, x(t-\lambda)\, d\lambda = \frac{1}{2} \cdot \left(\frac{1}{2}t + 1\right)(2-t) = \frac{1}{4}\left(4 - t^2\right)$$

IV. For $t > 2$, $h(\lambda) \cdot x(t-\lambda) = 0$. Therefore, $y(t) = 0$.

Then the convolution integral is as follows.

$$y(t) = \begin{cases} \frac{1}{4}(2+t)^2, & -2 \leq t < 0 \\ \frac{1}{4}\left(4 - t^2\right), & 0 \leq t < 2 \\ 0, & \text{otherwise} \end{cases}$$

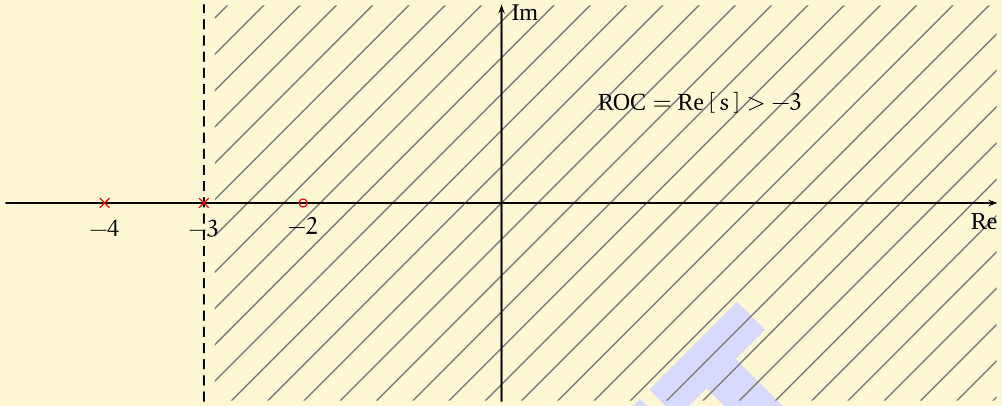


A5)

a) The transfer function has one zero and two poles.

$s = -2$, is zero

$s = -3$ and $s = -4$ are poles. Because the system is causal the ROC is right side of a vertical line which is bearing the rightmost pole.



b)

$$H(s) = \frac{s+2}{(s+3)(s+4)^2}$$
$$= \frac{A}{(s+3)} + \frac{B}{(s+4)} + \frac{C}{(s+4)^2}$$

$$A = \lim_{s\to -3} (s+3) H(s) = \lim_{s\to -3} \frac{s+2}{(s+4)^2} = -1$$
$$C = \lim_{s\to -4} (s+4)^2 H(s) = \lim_{s\to -4} \frac{s+2}{(s+3)} = 2$$
$$B = \lim_{s\to -4} \frac{d}{ds} (s+4)^2 H(s) = \frac{d}{ds} \lim_{s\to -4} \frac{s+2}{(s+3)}$$
$$= \lim_{s\to -4} \left(\frac{1}{(s+3)} - \frac{s+2}{(s+3)^2} \right) = 1$$

Therefore,

$$H(s) = -\frac{1}{(s+3)} + \frac{1}{(s+4)} + \frac{2}{(s+4)^2}$$

The inverse Laplace transform of the system function yields the impulse response of the system.

$$\mathcal{F}^{-1}\left[H(s)\right] = h(t) \; = -e^{-3t}u(t) + e^{-4t}u(t) + te^{-4t}u(t)$$

Answer all questions. Exam time : 150 minutes.

QUESTIONS

Q1) The following signals are given

$$x(t) = \begin{cases} -\frac{1}{2}t - 1, & -2 < t < 0 \\ 2t - 1, & 0 \le t < 1 \\ -t + 2, & 1 \le t < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$x[n] = -\delta[n+2] - 2\delta[n+1] - 3\delta[n] - 1.5\delta[n-1] + \delta[n-2] + 2\delta[n-3] + 0.5\delta[n-4]$$

a) Plot $x(t)$ and $x(-2t+4)$. Find energy of the signal. (8 p)

b) Plot $x[n]$ and $x[-n+2]$. Find energy of the signal. (9 p)

Q2) Find inverse transform of the following Fourier transforms.

a) Magnitude and phase of Fourier transform of a continuous time signal is given as (9 p)

$$M(\omega) = \frac{1}{\sqrt{\omega^2 + 1}}$$

$$\theta(\omega) = -\arctan(\omega)$$

b) Fourier transform of a discrete time signal is given as (8 p)

$$X(\omega) = 3e^{j3\omega} + 2e^{j2\omega} + e^{j\omega} - 1 - 2e^{-j\omega} - 3e^{-j2\omega}$$

Q3) Find impulse response of the following LTI and causal systems. Use time domain method.

a) (8 p)

$$y''(t) + 4y'(t) + 3y(t) = x(t)$$

Initial values of the impulse response are $y'(0^+) = 1$ and $y(0^+) = 0$.

b) (9 p)

$$y[n] - \frac{14}{15}y[n-1] + \frac{1}{5}y[n-2] = x[n]$$

Initial values of the impulse response are $y[0] = 1$ and $y[1] = \frac{14}{15}$.

Q4) Find

a) Bi-lateral Laplace transform of (8 p)

$$x(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{3t}u(-t)\ .$$

Do not forget to specify the ROC.

b) z-transform of (9 p)

$$x[n] = -\frac{5}{4}\left(\frac{1}{3}\right)^n u[n] + \frac{9}{4}\left(\frac{3}{5}\right)^n u[-1-n]\ .$$

Do not forget to specify the ROC.

Q5) In particular a system may or may not be

1. Memoryless
2. Time-invariant
3. Linear
4. Causal

5. Stable

Determine which of these properties hold which do not hold for each of the following signals. Justify your answer. In each example $y(t)$ or $y[n]$ denotes the system output, and $x(t)$ or $x[n]$ is the system input. (17 p)

(a) $y(t) = e^{x(t)}$

(b) $y(t) = x(t-1) - x(1-t)$

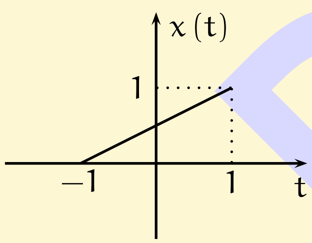
(c) $y(t) = [\sin(6t)]\, x(t)$

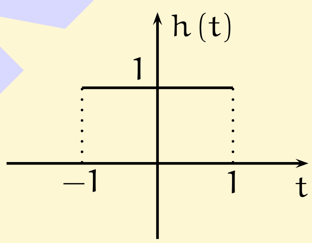
(d) $y[n] = x[n]\, x[n-1]$

(e) $y[n] = x[n-2] - x[n-17]$

(f) $y[n] = n x[n]$

Q6) Find convolution of the signals given below (15 p)





A.

$$\int (at+b)^2\, dt = \frac{1}{3a}(at+b)^3 + c$$

B.

$$e^{at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega - a}, \quad a < 0$$
$$-e^{at}u(-t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega - a}, \quad a > 0$$

C. $\mathcal{L}$ denotes bi-lateral Laplace transform. $\mathcal{UL}$ denotes uni-lateral Laplace transform.

$$e^{at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s - a}, \quad \mathrm{Re}[s] > a$$
$$-e^{at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s - a}, \quad \mathrm{Re}[s] < a$$

D.

$$a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1$$
$$-a^n u[-1-n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \quad |a| > 1$$

E. $\mathcal{Z}$ denotes bi-lateral z-transform. $\mathcal{UZ}$ denotes uni-lateral z-transform.

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad \mathrm{ROC} = |z| > |a|$$
$$-a^n u[-1-n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad \mathrm{ROC} = |z| < |a|$$

F. Solution of homogenous differential equation

$$y''(t) + ay'(t) + by(t) = 0$$

is

$$y(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

s_1 and s_2 are roots of the characteristic equation

$$D^2 + aD + b = 0$$

and constants C_1 and C_2 are determined from the initial conditions.

G. Solution of homogenous difference equation

$$y[n] + ay[n-1] + by[n-2] = 0$$

is

$$y[n] = C_1 \alpha_1^n + C_2 \alpha_2^n$$

α_1 and α_2 are roots of the characteristic equation

$$D^2 + aD + b = 0$$

and constants C_1 and C_2 are determined from the initial conditions.

H. Convolution integral

$$z(t) = \int_{-\infty}^{\infty} x(\lambda)\, y(t-\lambda)\, d\lambda$$

If $x(t)$ and $y(t)$ are time limited functions you should specify integral interval(s) where $x(\lambda)\, y(t-\lambda) \neq 0$.

I. Fourier transform for continuous-time signals

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

J. Fourier transform for discrete-time signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega$$

K. Bi-lateral Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \qquad x(t) = \frac{1}{2\pi j} \int_{a-\infty}^{a+\infty} X(s)\, e^{st} ds$$

L. z-transform for discrete-time signals

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad x[n] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z)\, z^{n-1} dz$$

M. Complex Fourier series for continuous-time signals

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, e^{-j\frac{2\pi}{T_0}kt} dt \qquad\qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt}$$

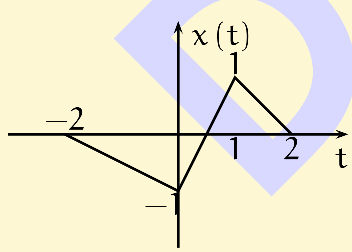
N. Complex Fourier series for discrete-time signals

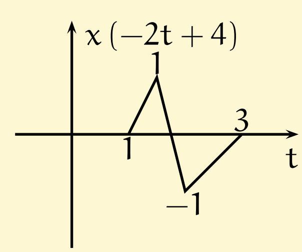
$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad\qquad x[n] = \sum_{k=k_0}^{k_0+N} a_k e^{j\frac{2\pi}{N}kn}$$

ANSWERS

A1)

a)





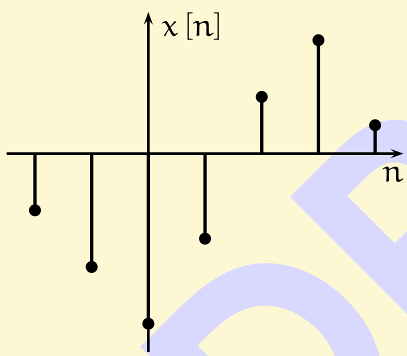
The value of $x(t)$ at time $t = t_a$; $x(t_a)$ appears at the transformed signal at time $t = 2 - \frac{t_a}{2} = t_b$

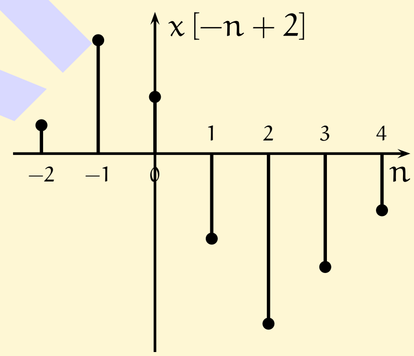
t_a	t_b
-2	3
0	2
1	1.5
2	1

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2(t)\,dt &= \int_{-2}^{0}\left(-\frac{1}{2}t-1\right)^2 dt+\int_{0}^{1}(2t-1)^2\,dt+\int_{1}^{2}(-t+2)^2\,dt \\
&= -\frac{2}{3}\left(-\frac{1}{2}t-1\right)^3\Bigg|_{-2}^{0}+\frac{1}{6}(2t-1)^3\Bigg|_{0}^{1}-\frac{1}{3}(-t+2)^3\Bigg|_{1}^{2} \\
&= \frac{4}{3}
\end{aligned}
$$

b)

The value of $x[n]$ at time $n = n_a$; $x[n_a]$ appears at the transformed signal at time $n = 2 - n_a =$





n_b

n_a	n_b
-2	4
-1	3
0	2
1	1
2	0
3	-1
4	-2

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} x^2[n] &= \sum_{n=-2}^{4} x^2[n] \\
&= (-1)^2 + (-2)^2 + (-3)^2 + (-1.5)^2 + 1^2 + 2^2 + 0.5^2 \\
&= 21.5
\end{aligned}
$$

A2)

a)

$$
\begin{aligned}
1 + j\omega &= \left(1+\omega^2\right)^{1/2} e^{j\,\arctan(\omega)} \\
(1+j\omega)^{-1} &= \left(1+\omega^2\right)^{-1/2} e^{-j\,\arctan(\omega)}
\end{aligned}
$$

$$
\frac{1}{j\omega+1} \xrightarrow{\mathcal{F}^{-1}} e^{-t}u(t)
$$

b)

$$
\begin{aligned}
X(\omega) &= 3e^{j3\omega} + 2e^{j2\omega} + e^{j\omega} - 1 - 2e^{-j\omega} - 3e^{-j2\omega} \\
&= 3e^{-j(-3)\omega} + 2e^{-j(-2)\omega} + e^{-j(-1)\omega} - 1\cdot e^{-j\,0\cdot\omega} - 2e^{-j\,1\cdot\omega} - 3e^{-j\,2\cdot\omega} \\
&= \sum_{n=-3}^{2} x[n]\, e^{-j\omega n}
\end{aligned}
$$

A3)

a) Impulse response is just solution of homogenous differential equation at $t > 0$. This is because impulse appears in the input at time $t = 0$ and specify the initial values of the system and

n	$x[n]$
-3	3
-2	2
-1	1
0	-1
1	-2
2	-3
elsewhere	0

disappears after. The characteristic equation of the system is as

$$D^2 + 4D + 3 = 0$$

The roots of the characteristic equation

$$(D+3)(D+1) = 0 \Rightarrow D_1 = -3,\ D_2 = -1$$

Then the impulse response

$$y(t) = C_1 e^{-3t} + C_2 e^{-t}$$

The constants are found from the initial values.

$$\begin{aligned} y(0^+) &= C_1 + C_2 = 0 \\ y'(0^+) &= -3C_1 - C_2 = 1 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Finally we get the impulse response

$$y(t) = -\frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

b)

Impulse response can be found by following similar approach to the above.

$$D^2 - \frac{14}{15}D + \frac{1}{5} = \left(D - \frac{3}{5}\right)\left(D - \frac{1}{3}\right) \Rightarrow D_1 = \frac{3}{5},\ D_2 = \frac{1}{3}$$

$$y[n] = C_1\left(\frac{3}{5}\right)^n + C_2\left(\frac{1}{3}\right)^n$$

$$\begin{aligned} y[0] &= C_1 + C_2 = 1 \\ y[1] &= C_1\frac{3}{5} + C_2\frac{1}{3} = \frac{14}{15} \end{aligned}$$

$$\begin{pmatrix} 1 & 1 \\ 3/5 & 1/3 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 14/15 \end{pmatrix} \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -\frac{15}{4}\begin{pmatrix} 1/3 & -1 \\ -3/5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 14/15 \end{pmatrix} \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 9 \\ -5 \end{pmatrix}$$

The impulse response is

$$y[n] = \frac{9}{4}\left(\frac{3}{5}\right)^n u[n] - \frac{5}{4}\left(\frac{1}{3}\right)^n u[n]$$

A4)

a)

$$
\begin{aligned}
e^{-t}u(t) &\xrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad \mathrm{Re}[s] > -1 \\
-e^{3t}u(-t) &\xrightarrow{\mathcal{L}} \frac{1}{s-3}, \quad \mathrm{Re}[s] < 3
\end{aligned}
$$

$$
\begin{aligned}
X(s) &= \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-3} \\
&= \frac{s-1}{(s+1)(s-3)}, \quad -1 < \mathrm{Re}[s] < 3
\end{aligned}
$$

b)

$$
\begin{aligned}
\left(\frac{1}{3}\right)^n u[n] &\xrightarrow{\mathcal{Z}} \frac{1}{1-\frac{1}{3}z^{-1}}, \quad \mathrm{ROC} = |z| > \frac{1}{3} \\
-\left(\frac{3}{5}\right)^n u[-1-n] &\xrightarrow{\mathcal{Z}} \frac{1}{1-\frac{3}{5}z^{-1}}, \quad \mathrm{ROC} = |z| < \frac{3}{5}
\end{aligned}
$$

$$
\begin{aligned}
X(z) &= -\frac{5}{4}\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{9}{4}\frac{1}{1-\frac{3}{5}z^{-1}} \\
&= -\frac{7}{2}\frac{1-\frac{3}{7}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{3}{5}z^{-1}\right)}, \quad \mathrm{ROC} = \frac{1}{3} < |z| < \frac{3}{5}
\end{aligned}
$$

A5) Definition of system properties

Memoryless or system with no memory: A system is said to be memoryless if output depends only present value of the input.

Time-invariance: A system is time invariant if it does not change its characteristics in time. It produces the same output to the same input at any time it is applied to the system.

Linearity: A system is linear if output is linearly dependent on the input. It does not depend on the exponential powers of the input.

Causality: A system is causal if it generates output only after input is applied.

Stability: A system is stable if output is bounded for a bounded input.

	Memoryless	Time-invariant	Linear	Causal	Stable
a	+	+	-	-	+
b	-	+	+	-	+
c	+	-	+	+	+
d	-	+	-	+	+
e	-	+	+	+	+
f	+	-	+	+	-

QUESTIONS

Q1) The following signals are given

$$x(t) = \begin{cases} t+1, & -1 \leq t < 0 \\ 1, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \\ 0, & 1\text{otherwise} \end{cases}$$

$$x[n] = -2\,\delta[n+3] + 3\,\delta[n+2] + 4\,\delta[n+1] - 5\,\delta[n] - 4\,\delta[n-1] + 2\,\delta[n-2] - 1\,\delta[n-3] + 2\,\delta[n-4]$$

a) Plot $x(t)$ and $x(-t/2 + 3/2)$. Compute energy of $x(t)$.

b) Plot $x[n]$ and $x[-n/2 + 3/2]$. Compute energy of $x[n]$.

Q2) Find Fourier transform of

a) $e^{-t}\cos(2t)\,u(t)$ and $te^{-2t}\,u(t)$,

b) $\left(\frac{1}{2}\right)^n \cos[2n]\,u[n]$ and $n\left(\frac{1}{2}\right)^n u[n]$

Q3)

a) Find convolution integral of the following continuous-time signals

$$e^{-2t}u(t) * e^{-t}u(t)$$

$$e^{-t}u(t) * e^{-t}u(t)$$

$$e^{-2t}u(t) * t\,e^{-t}u(t)$$

b) Find convolution sum of the following discrete-time signals

$$\left(\frac{1}{3}\right)^n u[n] \ * \ \left(\frac{1}{2}\right)^n u[n]$$
$$\left(\frac{1}{2}\right)^n u[n] \ * \ \left(\frac{1}{2}\right)^n u[n]$$
$$\left(\frac{1}{3}\right)^n u[n] \ * \ n\left(\frac{1}{2}\right)^n u[n]$$

Q4)

a) Transfer (system) function of an LTI and causal continuous-time system is given as in the following

$$H(s) = \frac{s+1}{(s+2)(s^2+4s+5)}$$

Find the impulse response.

b) Transfer (system) function of an LTI and causal discrete-time systems is given as in the following

$$H(z) = \frac{2z\left(12z^2 - 15z + 4\right)}{(2z-1)^2\ (3z-1)}$$

Find the impulse response.

Q5) A continuous-time LTI and causal system is given as in the following. The continuous-time signals are sampled with a sampling interval of $T = 0.05$ sec.

$$\frac{d}{dt}y(t) + 4y(t) = x(t)$$

Find the corresponding difference equation by approximating the differential equation. Use a) backward difference approximation, b) bilinear transform.

Q6) Find Fourier transform of the following signals.

a)

$$x(t) = \begin{cases} -1, & -1 \leq t < 0 \\ 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

b)

$$x(t) = e^{-t} \cos(2t)\, u(t)$$

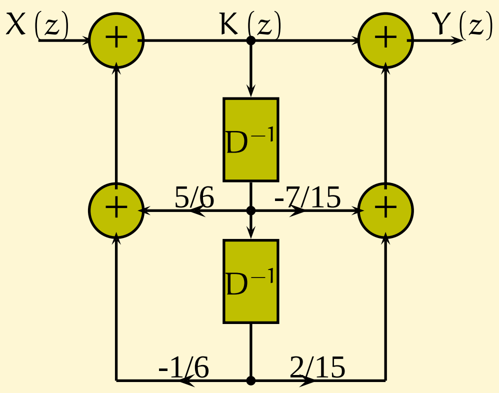
c)

$$x[n] = \begin{cases} -1, & n = -4, -3, -2, -1 \\ 1, & n = 1, 2, 3, 4 \\ 0, & \text{elsewhere} \end{cases}$$

d)

$$x[n] = \left(\frac{1}{3}\right)^{n} \cos[2n]\, u[n]$$

Q7) Block diagram of an LTI and causal system is given as in the following



a) Find the system function $H(z)$. b) Find the corresponding difference equation of the system.

Q8) Prove the following z-transform properties.

a)

$$\mathcal{Z}\left[\, x[n] * y[n] \,\right] = X(z)\, Y(z)\,, \quad \text{ROC} \subseteq (\, R_x \cap R_y \,)$$

b)

$$\mathcal{Z}\left[\sum_{k=-\infty}^{n} x[n]\right] = \frac{1}{1-z^{-1}} X(z)\,, \quad \text{ROC} \subseteq [\, R_x \cap (\, |z| > 1 \,)\,]$$

Hint:

$$\sum_{k=-\infty}^{n} x[n] = u[n] * x[n]$$

c)

$$\mathcal{Z}\left[\, a^n x[n] \,\right] = X\left(\frac{z}{a}\right), \quad \text{ROC} = |a|\, R_x$$

Q9) Determine properties (memoryless, causality, linearity, time-invariance, stability) of the following systems. Justify your answers clearly.

a)

$$\frac{d}{dt} y(t) + t y(t) = x(t)$$

b)

$$\frac{d}{dt}y(t) + 3y(t) + \cos(2t) = x(t)$$

c)

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

d)

$$y[n] - \frac{1}{2}y^2[n-1] = x[n]$$

I. INTEGRALS

$$\int (at+b)^2\, dt = \frac{1}{3a}(at+b)^3 + c$$

$$\int te^{at}\, dt = \frac{1}{a}te^{at} - \frac{1}{a^2}e^{at} + c$$

II. COMPLEX NUMBERS

Euler's Identity

$$e^{j\theta} = \cos(\theta) + j\ \sin(\theta)$$

Cauchy's integral

$$\oint_{\mathcal{C}} z^{-1}\, dz = 1$$

$$\oint_{\mathcal{C}} z^{n-1}\, dz = \delta[n]$$

Consider contour integral

$$\oint_{\mathcal{C}} F(z)\ dz$$

where

$$F(z) = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C_1}{z-c} + \frac{C_2}{(z-c)^2} + \cdots + \frac{C_n}{(z-c)^n}$$

Here, a, b, and c are poles of $F(z)$ and c is a multiple pole. A, B, and C_1 are called as residues of $F(z)$. The contour integral is equal to sum of these residues.

$$\oint_{\mathcal{C}} F(z)\ dz = A + B + C_1$$

III. FOURIER SERIES

Complex Fourier series for continuous-time signals.T_0 is the period.

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)\, e^{-j\frac{2\pi}{T_0}kt} dt \qquad\qquad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt}$$

Complex Fourier series for discrete-time signals. N is the period.

$$a_k = \frac{1}{N} \sum_{n=n_0}^{n_0+N} x[n]\, e^{-j\frac{2\pi}{N}kn} \qquad\qquad x[n] = \sum_{k=k_0}^{k_0+N} a_k e^{j\frac{2\pi}{N}kn}$$

IV. FOURIER TRANSFORMS

Fourier transform for continuous-time signals

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \qquad\qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

Fourier transform for discrete-time signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad\qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega$$

Fourier Transform Pairs.

$$
\begin{aligned}
e^{at}u(t) &\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega - a}, \quad a < 0 \\
-e^{at}u(-t) &\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega - a}, \quad a > 0
\end{aligned}
$$

$$
\begin{aligned}
a^{n}u[n] &\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1 \\
-a^{n}u[-1-n] &\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \quad |a| > 1
\end{aligned}
$$

The duality property:

$$
\begin{aligned}
y(t) &\overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) \\
z(t) = Y(t) &\overset{\mathcal{F}}{\longleftrightarrow} Z(\omega) = 2\pi y(-\omega)
\end{aligned}
$$

$\mathcal{L}$ denotes bi-lateral Laplace transform. $\mathcal{UL}$ denotes uni-lateral Laplace transform.

Bi-lateral Laplace transform

$$
X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st} dt \qquad\qquad x(t) = \frac{1}{2\pi j}\int_{a-\infty}^{a+\infty} X(s)\, e^{st} ds
$$

Laplace Transform Pairs.

$$e^{at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \mathrm{Re}[s] > a$$
$$-e^{at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}, \quad \mathrm{Re}[s] < a$$

VI. Z-TRANSFORM

$\mathcal{Z}$ denotes bi-lateral z-transform. $\mathcal{UZ}$ denotes uni-lateral z-transform.

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad\qquad x[n] = \frac{1}{2\pi j}\oint_{\mathcal{C}} X(z)\, z^{n-1} dz$$

z-Transform Pairs.

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad \mathrm{ROC} = |z| > |a|$$
$$-a^n u[-1-n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad \mathrm{ROC} = |z| < |a|$$

VII. LTI and causal systems

Solution of homogenous differential equation

$$y''(t) + ay'(t) + by(t) = 0$$

is

$$y(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

s_1 and s_2 are roots of the characteristic equation

$$D^2 + aD + b = 0$$

and constants C_1 and C_2 are determined from the initial conditions.

Solution of homogenous difference equation

$$y[n] + ay[n-1] + by[n-2] = 0$$

is

$$y[n] = C_1 \alpha_1^n + C_2 \alpha_2^n$$

α_1 and α_2 are roots of the characteristic equation

$$D^2 + aD + b = 0$$

and constants C_1 and C_2 are determined from the initial conditions.

Convolution integral

$$z(t) = \int_{-\infty}^{\infty} x(\lambda)\, y(t-\lambda)\, d\lambda$$

If $x(t)$ and $y(t)$ are time limited functions you should specify integral interval(s) where $x(\lambda)\, y(t-\lambda) \neq 0$.

Convolution sum

$$z[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$$

VIII. APPROXIMATION OF CONTINUOUS-TIME SYSTEMS BY DISCRETE-TIME SYSTEMS

Backward-difference approximation

$$s = \frac{1}{T}\left(1 - z^{-1}\right)$$

Bi-linear transform

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Frequency warping of bi-linear transform

$$\begin{aligned} \omega_t &= \frac{2}{T}\tan\left(\frac{\omega_n}{2}\right) \\ \omega_n &= 2\arctan\left(\frac{T}{2}\omega_t\right) \end{aligned}$$

Here, ω_t is angular frequency of continuous-time signal and ω_n is angular frequency variable of discrete-time signal.

$$\omega_n \approx T\omega_t, \quad \text{when, } \omega_t \ll \frac{2}{T}$$

QUESTIONS

Q1) (15 p)

a) Find the impulse response of the following linear time-invariant and casual system (by solving the homogenous equation in time domain).

$$D^2 y(t) + 2Dy(t) + 5y(t) = x(t) \ .$$

b) Find the frequency response of the following linear time-invariant and casual system.

$$y[n] - \frac{7}{6} y[n-1] + \frac{1}{3} y[n-2] = x[n] \ .$$

Q2) (15 p) A continuous time signal; $x(t) = u(t+1) - \frac{3}{2}u(t) + \frac{1}{2}u(t-2)$ is given.
a) Plot the signal. b) Find the energy of the signal. c) Plot $x\left(1 - \frac{t}{3}\right)$.

Q3) (20 p) Find (test and describe clearly) the properties (linearity, time-invariance, causality, memory, stability) of the following system. For testing stability and memory use the characteristic equation or find the impulse response.

$$y[n] - \frac{1}{2} y[n-1] = x[n] \ .$$

Q4) (15 p) Find the Fourier transform of the following signals.

a) $x(t) = u(t+1) - u(t-1)$, b) $x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t)$.

Q5) (14 p) Find the following convolutions.

a) $e^{-t}u(t) * e^{-2t}u(t)$, b) $\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n]$.

Q6) (15 p) Transfer function of an LTI and causal system and its partial fractions are given as

$$H(s) = \frac{s^2+6}{s^3-2s+4} = \frac{1}{s+2} + \frac{2}{s^2-2s+2} = \frac{1}{s+2} + \frac{j}{s-1+j} - \frac{j}{s-1-j}.$$

a) What is the ROC of $H(s)$? b) Is the system stable ? c) Find the impulse response.

Give your answers clearly.

Another transfer function:

$$H(s) = \frac{s^2-2}{s^3+2s+4} = \frac{1}{s+2} - \frac{2}{s^2+2s+2} = \frac{1}{s+2} + \frac{j}{s+1-j} - \frac{j}{s+1+j}.$$

Q7) (14 p)

a) From multiplication of two unit length complex numbers; $e^{ja} \cdot e^{jb}$, derive the trigonometric identities for $\cos(a+b)$ and $\sin(a+b)$ by using the Euler's identity for each complex number.

$$e^{ja} \cdot e^{jb} = e^{j(a+b)}.$$

d) Show that $y(t) = A\,e^{-t}\cos(t+\theta) + (1-t)^2$ is a solution for the following differential equation. A and θ are determined by auxiliary conditions on $y(t)$.

$$D^2y(t) + 2Dy(t) + 2y(t) = 2t^2.$$

c) Find polar representations of the following complex numbers.

$$\frac{1}{2} + j\frac{\sqrt{3}}{2}, \qquad 0 + j, \qquad 1 + 0j, \qquad \cos(2\pi f_o t) + j\sin(2\pi f_o t).$$

ANSWERS

A1)

a) The characteristic equation of the differential equation is

$$s^2 + 2s + 5 = 0\,.$$

The roots of the characteristic equation

$$(s+1)^2 + 4 = 0 \;\Rightarrow\; (s+1)^2 = -4$$
$$(s+1) = \mp j2$$
$$s = -1 \mp j2\,.$$

$$s_1 = -1 - j2$$
$$s_2 = -1 + j2\,.$$

The impulse response is then,

$$\begin{aligned} y(t) &= C_1 e^{s_1 t} + C_2 e^{s_2 t}, \qquad \text{for} \quad t > 0 \\ y(t) &= 0, \qquad \text{for} \quad t < 0\,. \end{aligned}$$

The constants C_1 and C_2 should be determined from the initial conditions. Integrating the differential equation one and twice and evaluate the integrals in the interval of, $0^- \leq t \leq 0^+$ we obtain

$y(0^+)$ and $y'(0^+)$.

$$\begin{aligned} Dy(t) + 2y(t) + 5D^{-1}y(t) &= u(t) \\ y(t) + 2D^{-1}y(t) + 5D^{-2}y(t) &= tu(t) \end{aligned}$$

Evaluating the definite integrals in the second equation over $[0^-,\ 0^+]$

$$y(t)\Big|_{0^-}^{0^+} + 2D^{-1}y(t)\Big|_{0^-}^{0^+} + 5D^{-2}y(t)\Big|_{0^-}^{0^+} = tu(t)\Big|_{0^-}^{0^+}$$

we get $y(0^+) = 0$. We assume that the solution (impulse response) does not have any impulse at the origin.

$$\begin{aligned} D^{-1}y(t)\Big|_{0^-}^{0^+} &\approx 0 \\ D^{-2}y(t)\Big|_{0^-}^{0^+} &\approx 0\,. \end{aligned}$$

We also employ the causality property of the system: $y(0^-) = 0$ and $y'(0^-) = 0$. Likewise, evaluating the first equation over $[0^-,\ 0^+]$

$$Dy(t)\Big|_{0^-}^{0^+} + 2y(t)\Big|_{0^-}^{0^+} + 5D^{-1}y(t)\Big|_{0^-}^{0^+} = u(t)\Big|_{0^-}^{0^+}$$

yields $y'(0^+) = 1$. Then,

$$\begin{aligned} C_1 + C_2 &= 0 \\ s_1 \cdot C_1 + s_2 \cdot C_2 &= 1 \end{aligned}$$

The coefficients are computed as

$$\begin{aligned} C_1 &= \frac{1}{s_1 - s_2} = -\frac{1}{j4} \\ C_2 &= \frac{1}{s_2 - s_1} = \frac{1}{j4}\,. \end{aligned}$$

We replace these coefficients in the solution

$$\begin{aligned} y(t) &= -\frac{1}{j4}e^{(-1-j2)t} + \frac{1}{j4}e^{(-1+j2)t} \\ &= \frac{1}{2}e^{-t}\left(-\frac{1}{j2}e^{-j2t} + \frac{1}{j2}e^{j2t}\right) = \frac{1}{2}e^{-t}\sin(2t) \end{aligned}$$

The solution is then,

$$y(t) = \frac{1}{2}e^{-t}\sin(2t)\,u(t)\,.$$

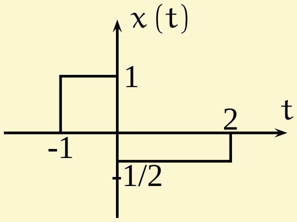
b) We obtain Fourier transform of the both sides of the difference equation. To do this we employ shifting property of the Fourier transform: $\mathcal{F}[x[n-k]] = e^{-j\Omega k}X(\Omega)$.

$$\begin{aligned} Y(\Omega) - \frac{7}{6}e^{-j\Omega}Y(\Omega) + \frac{1}{3}e^{-2j\Omega}Y(\Omega) &= X(\Omega) \\ Y(\Omega)\left(1 - \frac{7}{6}e^{-j\Omega} + \frac{1}{3}e^{-2j\Omega}\right) &= X(\Omega) \end{aligned}$$

$$\frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{7}{6}e^{-j\Omega} + \frac{1}{3}e^{-2j\Omega}} = H(\Omega)\,.$$

A2)

a)



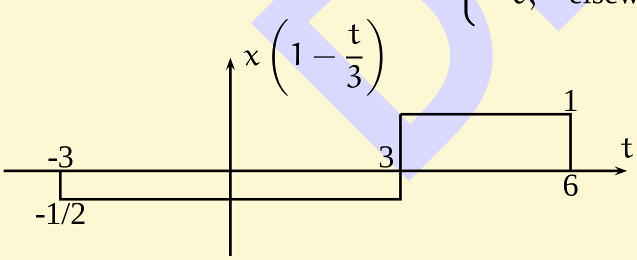
b)

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-1}^{0} |1|^2 \, dt + \int_{0}^{2} \left|-\frac{1}{2}\right|^2 dt = 1 \cdot 1 + \frac{1}{4} \cdot 2 = \frac{3}{2} \quad J.$$

c)

$$x(t) = \begin{cases} 1, & -1 \le t < 0, \\ -\frac{1}{2}, & 0 \le t < 2, \\ 0, & \text{elsewhere}. \end{cases}$$

$$x\left(1 - \frac{t}{3}\right) = \begin{cases} 1, & -1 \le 1 - \frac{t}{3} < 0, \rightarrow \; 3 < t \le 6 \\ -\frac{1}{2}, & 0 \le 1 - \frac{t}{3} < 2, \rightarrow \; -3 < t \le 3 \\ 0, & \text{elsewhere}. \end{cases}$$



A3)

Linearity:

I. Additivity

Input		Output
$x_1(t)$	$\longrightarrow$	$y_1(t)$
$x_2(t)$	$\longrightarrow$	$y_2(t)$
$x_1(t)+x_2(t)$	$\longrightarrow$	$y_1(t)+y_2(t)$

$$y_1[n]-\frac{1}{2}y_1[n-1] = x_1[n]$$

$$y_2[n]-\frac{1}{2}y_2[n-1] = x_2[n]$$

+

$$(y_1[n]+y_2[n])-\frac{1}{2}(y_1[n-1]+y_2[n-1]) = (x_1[n]+x_2[n])$$

II. Homogeneity

Input		Output
$x(t)$	$\longrightarrow$	$y(t)$
$\alpha x(t)$	$\longrightarrow$	$\alpha y(t)$

$$\begin{aligned}
\alpha\left(y[n]-\frac{1}{2}y[n-1]\right) &= \alpha x[n] \\
\alpha y[n]-\alpha\frac{1}{2}y[n-1] &= \alpha x[n] \\
(\alpha y[n])-\frac{1}{2}(\alpha y[n-1]) &= (\alpha x[n])
\end{aligned}$$

Since both additivity and homogeneity are satisfied the system is linear.

Time-invariance: The coefficients of the difference equitation does not change with time; they are constant. Therefore the system is time-invariant.

Causality: It is known that output is zero for zero input for a linear linear system. A causal system produces an output only when an input is applied; output is zero when the system is at rest. Recall the definition of the causality,

$$x[n] = \begin{cases} 0, & n < k \\ x_1[n], & n \geq k \end{cases} \qquad y[n] = \begin{cases} 0, & n < k \\ y_1[n], & n \geq k \end{cases}$$

From the definition of the causality it is seen that if the system is linear (satisfies zero output for zero input) the system has to be causal too. In other words initial conditions of the difference equation are all zero ($y[n] = 0$ for all $n < k$).

Memory : The system needs present value of the input ($x[n]$) and past value of the output ($y[n-1]$) to generate its output.

$$y[n] = \frac{1}{2}y[n-1] + x[n] \ .$$

The system should remember the past value; $y[n-1]$ at present time n. Therefore the system is with memory.

BIBO Stability: If $|x[n]| < A, \quad |y[n]| < B$. Here, $\infty > A,\ B > 0$.

$$\left| y[n] - \frac{1}{2} y[n-1] \right| = |x[n]|$$
$$|y[n]| + \frac{1}{2} |y[n-1]| \le |x[n]|$$
$$B + \frac{1}{2} B \le A$$
$$B \le \frac{2}{3} A$$

For bounded input $(0 < A < \infty)$, the output is also bounded $(0 < B \le \frac{2}{3} A < \infty)$. The system is stable.

A4)

a)

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \\ &= \int_{-1}^{1} 1 \cdot e^{-j\omega t} dt \\ &= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-1}^{1} = -\frac{1}{j\omega} e^{-j\omega} + \frac{1}{j\omega} e^{j\omega} \\ &= 2\frac{\sin(\omega)}{\omega}. \end{aligned}$$

Or,

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}[u(t+1)-u(t-1)] \;=\; \mathcal{F}[u(t+1)]-\mathcal{F}[u(t-1)] \\
&= \left(\frac{1}{j\omega}+\pi\delta(\omega)\right)e^{j\omega}-\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)e^{-j\omega} \\
&= \frac{1}{j\omega}e^{j\omega}-\frac{1}{j\omega}e^{-j\omega}+\pi\delta(\omega)\,e^{j\omega}-\pi\delta(\omega)\,e^{-j\omega} \\
&= 2\frac{\sin(\omega)}{\omega}+\pi\delta(\omega)\,e^{j0}-\pi\delta(\omega)\,e^{-j0} \\
&= 2\frac{\sin(\omega)}{\omega}.
\end{aligned}$$

b)

$$\begin{aligned}
\mathcal{F}[x(t)] &= \mathcal{F}\left[e^{-t}u(t)+e^{t}u(-t)\right] \;=\; \mathcal{F}\left[e^{-t}u(t)\right]+\mathcal{F}\left[e^{t}u(-t)\right] \\
&= \frac{1}{1+jw}+\frac{1}{1-jw} \;=\; \frac{2}{1+\omega^2}
\end{aligned}$$

A5)

a)

$$e^{-t}\,u(t)\,*\,e^{-2t}\,u(t)=\int_{-\infty}^{\infty} e^{-\lambda}\,u(\lambda)\,e^{-2(t-\lambda)}\,u(t-\lambda)\,d\lambda$$

$$u(\lambda)\,u(t-\lambda)=\begin{cases}1, & t\geq 0 \text{ and } 0\leq\lambda\leq t\\ 0, & t<0\,.\end{cases}$$

$$\int_{-\infty}^{\infty} e^{-\lambda} u(\lambda) e^{-2(t-\lambda)} u(t-\lambda) d\lambda = \int_0^t e^{-\lambda} e^{-2(t-\lambda)} d\lambda = e^{-2t} \int_0^t e^{\lambda} d\lambda$$

$$= e^{-2t} e^{\lambda} \Big|_0^t = e^{-2t} \left(e^t - 1\right)$$

$$= e^{-t} - e^{-2t}.$$

For all t,

$$e^{-t} u(t) * e^{-2t} u(t) = e^{-t} u(t) - e^{-2t} u(t)\ .$$

b)

$$\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \left(\frac{1}{3}\right)^{n-k} u[n-k]$$

$$u[k]\, u[n-k] = \begin{cases} 1, & n \geq 0 \text{ and } 0 \leq k \leq n \\ 0, & n < 0\,. \end{cases}$$

$$\sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \left(\frac{1}{3}\right)^{n-k} u[n-k] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k} = \left(\frac{1}{3}\right)^n \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k$$

$$= \left(\frac{1}{3}\right)^n \frac{1 - \left(\frac{3}{2}\right)^{n+1}}{1 - \frac{3}{2}} = -2 \left(\frac{1}{3}\right)^n \left(1 - \left(\frac{3}{2}\right)^{n+1}\right)$$

$$= 3 \left(\frac{1}{2}\right)^n - 2 \left(\frac{1}{3}\right)^n.$$

For all n,

$$\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{3}\right)^n u[n] = 3 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{3}\right)^n u[n]\ .$$

A6)

a) The poles of the transfer function are $s = 1 - j$ and $s = 1 + j$. Since the system is causal the

ROC is right side of the vertical line containing the rightmost pole(s) of the system function. Then the ROC is $\mathrm{Re}\,[s] > 1$.

b) The system is unstable because the ROC does not contain $j\omega$ axis.

c)

$$\begin{aligned} h\,(t) &= j\,e^{(1-j)t}\,u\,(t) - j\,e^{(1+j)t}\,u\,(t) \\ &= e^{t}\left(j\,e^{-jt} - j\,e^{jt}\right)\,u\,(t) \\ &= 2e^{t}\,\sin\,(t)\,u\,(t)\ . \end{aligned}$$

$h\,(t)$ grows as t increases. This is because the system is unstable. If the poles were located at the left half plane of the complex plane $h\,(t)$ would be decreasing; the system was stable.

A7)

a)

$$\begin{aligned} e^{ja}\,e^{jb} &= (\,\cos\,(a) + j\sin\,(a)\,)\ (\,\cos\,(b) + j\sin\,(b)\,) \\ &= \cos\,(a)\cos\,(b) - \sin\,(a)\sin\,(b) + j\,(\,\sin\,(a)\cos\,(b) + \sin\,(b)\cos\,(a)\,)\ . \end{aligned}$$

$$\begin{aligned} e^{ja}\,e^{jb} &= e^{j(a+b)} \\ &= \cos\,(a+b) + j\sin\,(a+b)\ . \end{aligned}$$

Therefore,

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b)\\ \sin(a+b) &= \sin(a)\cos(b) + \sin(b)\cos(a)\ .\end{aligned}$$

b)

$$\begin{aligned}y(t) &= A\,e^{-t}\,\cos(t+\theta) + (1-t)^2\\ Dy(t) &= -A\,e^{-t}\,\sin(t+\theta) - A\,e^{-t}\,\cos(t+\theta) - 2(1-t)\\ D^2y(t) &= 2A\,e^{-t}\,\sin(t+\theta) + 2\\ \\ D^2y(t) + 2Dy(t) + 2y(t) &= 2t^2\end{aligned}$$

c)

$$\begin{aligned}\frac{1}{2} + j\frac{\sqrt{3}}{2} &= e^{j\pi/3}\\ 0 + j &= e^{j\pi/2}\\ 1 + 0j &= e^{j\,0}\\ \cos(2\pi f_o t) + j\sin(2\pi f_o t) &= e^{j2\pi f_o t}\,.\end{aligned}$$

Answer all the questions. Q1, Q6, Q7 are 10p, others are 15p.

Exam time : 90 min.

QUESTIONS

Q1) Find the frequency response of the following linear time-invariant and casual system.

$$D^2 y(t) + 5Dy(t) + 6y(t) = x(t) \,.$$

Q2) Output of an LTI and causal system is $y[n] = 2\left(\frac{2}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^n u[n]$ for input $x[n] = \left(\frac{2}{3}\right)^n u[n]$. Find the impulse response of the system.

Q3) Plot continuous time signal; $x(t) = (t+1)\,u(t+1) - 4u(t-1) - (t-3)\,u(t-3)$. (Find the behavior of $x(t)$ for each time interval of $t < -1, \quad -1 < t < 1, \quad 1 < t < 3, \quad$ and $t > 3$).

Q4) Find the Fourier transform of $x(t) = te^{-t}u(t)$. Use the following Fourier transform property.

$$\mathcal{F}[t\,x(t)] = j\frac{d}{d\omega}X(\omega) \,.$$

Q5) Find the Fourier transform of $x[n] = \left(\frac{1}{2}\right)^{|n|} = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$. Recall that:

$$\sum_{n=-\infty}^{-1} \alpha^n = \frac{1}{\alpha - 1}, \quad |\alpha| > 1$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha}, \quad |\alpha| < 1 \,.$$

Q6) Find the convolution; $2t\,e^{-2t}u(t) * e^{-2t}u(t)$.

Q7) Find the convolution; $n\left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^n u[n]$. Remember that

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2}.$$

Q8) Transfer function of an LTI and causal system and its partial fractions are given as

$$H(s) = \frac{1}{s^3+4s^2+5s+2} = \frac{1}{s+2} - \frac{s}{s^2+2s+1} = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}.$$

a) What is the ROC of $H(s)$? b) Is the system stable ? c) Find the impulse response.

Give your answers shortly and clearly. Recall that

$$\mathcal{L}\left[te^{-at}u(t)\right] = \frac{1}{(s+a)^2}, \quad \text{Re}[s] > -a.$$

Q9) Transfer function of an LTI and causal system and its partial fractions are given as

$$H(z) = \frac{z^2}{z^2 - \frac{7}{6}z + \frac{1}{3}} = \frac{4z}{z-\frac{2}{3}} - \frac{3z}{z-\frac{1}{2}}.$$

a) What is the ROC of $H(z)$? b) Is the system stable ? c) Find the impulse response.

Give your answers shortly and clearly.

Some other questions

Q1) Impulse response of an LTI and causal system with a transfer function of

$$H(s) = \frac{s+a}{(s+a)^2 + b^2}$$

is

$$h(t) = \mathcal{L}^{-1}[H(s)] = e^{-at}\cos(b)\,u(t)\ .$$

Explain shortly and clearly where and why the poles of the transfer should be located in ? Use the absolute integral of the impulse response;

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{0}^{\infty} e^{-at}|\cos(b)|\,dt \le \int_{0}^{\infty} e^{-at}dt\,.$$

and/or, $\lim\limits_{t\to\infty} h(t)$.

Q2) An LTI causal system is given as

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]\ .$$

Transfer function of this system is

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = -\frac{2}{3} + \frac{5}{3}\frac{1}{1-\frac{1}{2}z^{-1}}\,.$$

And the impulse response is

$$h[n] = -\frac{2}{3}\delta[n] + \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]\ .$$

The inverse system of this system is $G(z) = 1/H(z)$. a) Find difference equation of the inverse system. b) Find the impulse response of the inverse system.

Q3) Find the residues of the following functions.

$$F_1(z) = \frac{z^2 - 2}{z^3 - 4z^2 + 5z - 2}$$

and

$$F_2(z) = \frac{z}{z^2 - 3z + 2}$$

Q4) Find the following limit.

$$\lim_{t \to \infty} e^{(a+jb)\,t}$$

Q5) Find the solution of the following differential equation.

$$Dy(t) + 2y(t) = (-2\sin(t) + 2\cos(t))\,u(t), \qquad y(0^+) = 0.$$

Q6) Find the solution of the following difference equation.

$$y[n] - \frac{1}{2}y[n-1] = \frac{1}{3}y[n]\,u[n], \qquad y[0] = 1.$$

Q7) Find the the following sum.

$$\sum_{n=0}^{3} \left(\sqrt{2} + j\sqrt{2}\right)^n$$