Complex Analysis

Parametric interval			Curve
$0 \leq t \leq 1$			$z\,(t)$
$0 \leq t \leq \alpha$	$\rightarrow$	$0 \leq \dfrac{t}{\alpha} \leq 1$	$z\left(\dfrac{t}{\alpha}\right)$
$a \leq t \leq a+\alpha$	$\rightarrow$	$0 \leq t-a \leq \alpha$	$z\left(\dfrac{t-a}{\alpha}\right)$
		$0 \leq \dfrac{t-a}{\alpha} \leq 1$	

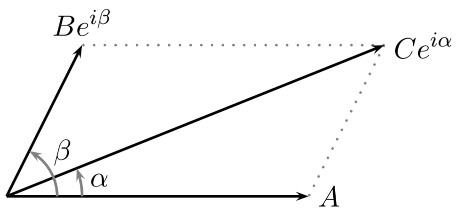
$$z = z\,(t) \quad \rightarrow \quad \gamma \quad a \leq t \leq b$$

$$z = z\,(-t) \quad \rightarrow \quad -\gamma \quad -b \leq t \leq -a$$

$$\begin{aligned} \Gamma &= (\,\gamma_1,\, \gamma_2,\, \ldots,\, \gamma_{n-1},\, \gamma_n\,) & a \leq t \leq b \\ -\Gamma &= (\,-\gamma_n,\, -\gamma_{n-1},\, \ldots,\, -\gamma_2,\, -\gamma_1) & -b \leq t \leq -a \end{aligned}$$

Sum of two trigonometric functions can be obtained by using complex exponentials (phasors). Consider sum of two cosines. Real part of an complex exponential is a cosine. Therefore real part of sum of complex exponentials corresponding to these cosines provide the sum of the two cosines.

$$\begin{aligned} A\cos(\theta) + B\cos(\theta+\beta) &= C\cos(\theta+\alpha) \\ Ae^{i\theta} + Be^{i(\theta+\beta)} &= Ce^{i(\theta+\alpha)} \\ e^{i\theta}\left(A + Be^{i\beta}\right) &= Ce^{i\alpha}\, e^{i\theta} \\ A + Be^{i\beta} &= Ce^{i\alpha} \end{aligned}$$



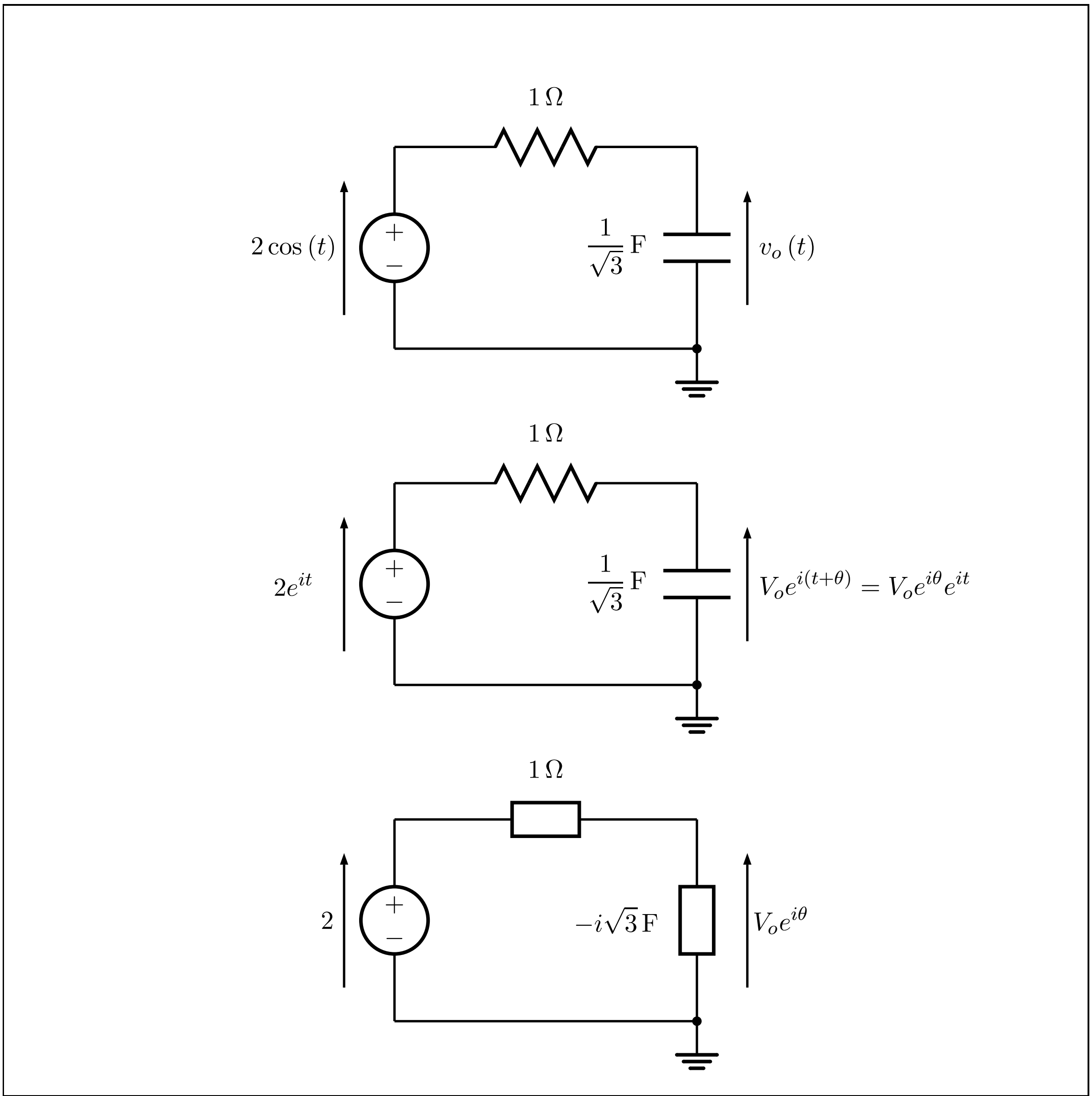
Example : Find $3\cos(\theta) + 4\sin(\theta)$.

$$\begin{aligned} 3 + 4e^{-i\pi/2} &= 3 - 4i &= 5e^{-i0.93} \\ 3\cos(\theta) + 4\sin(\theta) &= 5\cos(\theta - 0.93) \end{aligned}$$

Example :
In the following circuit (upper circuit) AC source; $2\cos(t)$ is connected for a long time and the responses to this source are at steady state. Since $\cos(t) = \text{Re}\left[e^{it}\right]$, the voltage on capacitor for AC source is real part of the output on the capacitor for the complex exponential source; $v_o(t) = \text{Re}\left[V_o e^{i\theta} e^{it}\right]$ (middle circuit). The voltage-current ratio (impedance) of the R and C for the exponential excitation are constant and $1\,\Omega$ and $-i\sqrt{3}\,\Omega$ respectively. Consequently the circuit is reduced to a DC resistance circuit (lower circuit). The circuit is a simple voltage divider and the capacitor voltage is $V_o e^{i\theta} = \dfrac{2}{1 - i\sqrt{3}} \cdot \left(-i\sqrt{3}\right)$. Using this methodology the steady state voltage $v_o(t)$ on the capacitor can be easily obtained.

$$\begin{aligned} V_o e^{i\theta} &= \frac{2}{1 - i\sqrt{3}} \cdot \left(-i\sqrt{3}\right) &= 2\,\frac{-i\sqrt{3}\left(3 - i\sqrt{3}\right)}{1 + 3} \\ &= \frac{1}{2}\left(-i\sqrt{3} + 3\right) &= \sqrt{3}e^{-i\pi/6} \end{aligned}$$

$$\begin{aligned} v_o(t) &= \text{Re}\left[V_o e^{i\theta} e^{it}\right] &= \text{Re}\left[\sqrt{3}e^{-i\pi/6} e^{it}\right] \\ &= \text{Re}\left[\sqrt{3}e^{i(t-\pi/6)}\right] &= \sqrt{3}\cos(t - \pi/6) \quad \text{Volt} \end{aligned}$$



A list of definitions and theorems from the textbook ”Fundamentals of Complex Analysis for Mathematics, Science and Engineering” by E. B. Saff and A. D. Snider.

Definition: Let $f(z)$ be a function defined in the neighborhood of z_0. Then $f(z)$ is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Definition: Let $f(z)$ be a complex-valued function defined in the neighborhood of z_0. Then the derivative of $f(z)$ at z_0 is given by

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z)}{\Delta z},$$

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provided this limit exists.

Definition: A complex-valued function $f(z)$ is said to be analytic on an open set G if it has a derivative at every point of G.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be defined in some open set G. If the first partial derivatives of u and v are continuous and satisfy the Cauchy-Riemann equations at all points of G, then $f(z)$ is analytic in G.

Definition: A real-valued function $\phi(x,y)$ is said to be harmonic in a domain D if all its second-order partial derivatives are continuous in D and if at each point of D, ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

Theorem: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then each of the functions $u(x,y)$ and $v(x,y)$are harmonic in D.

De Moivre's formula

$$[\,r\cos(\theta) + i\,r\sin(\theta)\,]^n = r^n\cos(n\theta) + i\,r^n\sin(n\theta)$$

Elementary functions.

The complex exponential function.
Definition: If $z = x + iy$, then e^z is defined to be a complex number

$$e^z = e^x(\cos(y) + i\sin(y)).$$

Definition: Given any complex number z, we define

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}.$$

Definition: For any complex number z we define

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh(z) = \frac{e^{z} + e^{-z}}{2}.$$

The logarithmic function.
Definition: If $z \neq 0$, then we define $\log(z)$ to any of the infinitely many values

$$\log(z) = \mathrm{Log}(|z|) + i(\theta + 2\pi k), \qquad k \in \mathbb{Z},$$

where θ denotes a particular value of $\arg(z)$.
The branch of the logarithm for $k = 0$ is called the principal value of $\log(z)$ and we refer it as the principal value of $\log(z)$. We denote this function by $\mathrm{Log}(z)$, i.e.,

$$\mathrm{Log}(z) = \mathrm{Log}(|z|) + i\mathrm{Arg}(z).$$

Definition: If α is a complex constant and $z \neq 0$, then we define z^α by

$$z^\alpha = e^{\alpha\log(z)}$$

If α is not a real rational number, we obtain infinitely many different values for z^α. On the other hand, if $\alpha = m/n$ where m and $n > 0$ are integers having no common factor, then n distinct values of $z^{m/n}$, namely

$$z^{m/n} = e^{(m/n)\mathrm{Log}(|z|)} e^{i(m/n)(\mathrm{Arg}(z)+2\pi k)} \qquad (k = 0, 1, \dots, n-1).$$

Definition: A point set γ in the complex plane is said to be a smooth arc if it is the range of some continuous complex-valued function $z = z(t)$, $a \le t \le b$, which satisfies the following conditions:

(i) $z(t)$ has a continuous derivative on $[a, b]$,

(ii) $z'(t)$ never vanishes on $[a, b]$,

(iii) $z(t)$ is one-to-one on $[a, b]$.

A point set γ is called a smooth closed curve if it is the range of continuous function $z = z(t)$, $a \le t \le b$, satisfying conditions (i) and (ii) above and the following:

(iii)′ $z(t)$ is one-to-one on the half-open interval $[a, b)$, but $z(b) = z(a)$ and $z'(b) = z'(a)$.

The phrase "γ is a smooth curve" means that γ is either a smooth arc or a smooth closed curve.

Definition: A contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \dots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \dots, n-1$.

Γ is said to be a closed contour or a loop if its initial and terminal points coincide. A simple closed contour is a closed contour with no multiple points other than its initial-terminal point; in other words, if $z = z(t)$, $a \le t \le b$, is a parametrization of the closed contour, then $z(t)$ is one-to-one on the half-open interval $[a, b)$.

Theorem: A simple closed contour separates the plane into two domains, each having the contour as its boundary. One of these domains, called the "interior", is bounded; the other called the "exterior", is unbounded.

The direction along Γ can be completely specified by declaring its initial-terminal point and stating which domain (interior or exterior) lies to the left, we say that Γ is positively oriented (counterclockwise direction). Otherwise Γ is said to oriented negatively (clockwise direction).

The length of a smooth curve $\gamma : z = z(t)$, $a \le t \le b$.

$$l(\gamma) = \int_a^b z'(t)\, dt$$

The contour integral.

Consider a function $f(z)$ which is defined over a directed smooth curve γ with initial point α and terminal point β.

For any positive integer n, we define a partition $\mathcal{P}_n$ of γ to be a finite number of points $z_0, z_1, \ldots, z_n$ on γ such that $z_0 = \alpha$, $z_n = \beta$, and z_{k-1} precedes z_k on γ for $k = 1, 2, \ldots, n$. If we compute the arc length along γ between every consecutive pair of points (z_{k-1}, z_k), the largest of these lengths provide a measure of "fitness" of the subdivision; the maximum length is called the mesh the partition and is denoted by $\mu(\mathcal{P}_n)$.
Now let $c_1, c_2, \ldots, c_n$ be any points of γ such that c_1 lies on the arc from z_0 to z_1, c_2 lies on the arc from z_1 to z_2, etc. Under these circumstances the sum $S(\mathcal{P}_n)$ defined by

$$S(\mathcal{P}_n) = f(c_1)(z_1 - z_0) + f(c_2)(z_2 - z_1) + \cdots + f(c_n)(z_n - z_{n-1})$$

is called Riemann sum for the function f corresponding to the partition $\mathcal{P}_n$. On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S(\mathcal{P}_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\,\Delta z_k.$$

Definition: Let $f(z)$ be a complex-valued function defined on the directed smooth curve γ. We say that $f(z)$ is integrable along γ If there exist a complex number L which the limit of every sequence of Riemann sums $S(\mathcal{P}_1), S(\mathcal{P}_2), \ldots, S(\mathcal{P}_n), \ldots$ corresponding to any sequence of partitions of γ satisfying $\lim\limits_{n\to\infty} \mu(\mathcal{P}_n) = 0$; i.e.,

$$\lim_{n\to\infty} S(\mathcal{P}_n) = L \quad \text{whenever} \quad \lim_{n\to\infty} \mu(\mathcal{P}_n) = 0.$$

The constant L is called integral of $f(z)$ along γ, and we write

$$L = \int_\gamma f(z)\,dz \quad \text{or} \quad L = \int_\gamma f.$$

Theorem: If $f(z)$ is continuous on the directed smooth curve γ, then $f(z)$ is integrable along γ.

Theorem: If the complex-valued function $f(t)$ is continuous on $[a, b]$ and $F'(t) = f(t)$ for all t in $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Theorem: Let $f(z)$ be a function continuous on the directed smooth curve γ. Then if $z = z(t)$, $a \leq t \leq b$, is any admissible parametrization of γ consistent with its direction, we have

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt \left(= \int_a^b f(z(t))\,\frac{dz}{dt}(t)\,dt \right).$$

Corollary: If $f(z)$ is continuous on the directed smooth curve γ and if $z = z_1(t)$, $a \leq t \leq b$, and $z = z_2(t)$, $c \leq t \leq d$, are any two admissible parameterizations of γ consistent with its direction, then

$$\int_a^b f(z_1(t))\,z_1'(t)\,dt = \int_c^d f(z_2(t))\,z_2'(t)\,dt.$$

Definition: Suppose that Γ is a contour consisting of the directed smooth curves $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, and let $f(z)$ be a function continuous on Γ. Then the contour integral of $f(z)$ along Γ is denoted by the symbol $\int_\Gamma f(z)\,dz$ and is defined by the equation

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz.$$

Theorem: If $f(z)$ is continuous on the contour Γ and if $|f(z)| \leq M$ for all z on Γ, then

$$\left|\int_\Gamma f(z)\,dz\right| \leq Ml(\Gamma).$$

Theorem: Suppose that the function $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$ throughout D; i.e., $dF(z)/dz = f(z)$ at each z in D ($F(z)$ is analytic in D). Then for any contour Γ lying in D, with initial point z_I and terminal point z_T, we have

$$\int_\Gamma f(z)\,dz = F(z_T) - F(z_I).$$

Corollary: If $f(z)$ is continuous in a domain D and has an antiderivative $F(z)$ throughout D, then

$$\int_\Gamma f(z)\,dz = 0$$

for all loops Γ lying in D.

Definition: A simply connected domain D is a domain having the following property: If Γ is any simple closed contour lying in D, then the domain interior to Γ lies wholly in D.

Theorem (Green's Theorem; Curl Theorem in the Plane): Let $\mathbf{V} = (V_1, V_2)$ be a continuously differentiable vector field defined on a simply connected domain D, and let Γ be a positively oriented simple closed contour in D. Then the line integral of $\mathbf{V}$ around Γ equals the integral of $(\partial V_2/\partial x - \partial V_1/\partial y)$, integrated with respect to area over the domain D' interior to Γ; i.e.,

$$\int_\Gamma (V_1 dx + V_2 dy) = \iint\limits_{D'} \left(\frac{\partial V_2}{dx} - \frac{\partial V_1}{dy}\right) dxdy.$$

The left hand side of this equation is the work done by the force; $\mathbf{V} = V_1(x,y) + iV_2(x,y)$ traversing the closed contour Γ which is the boundary of the surface D'.

Theorem (Cauchy's Integral Theorem): If $f(z)$ is analytic in a simply connected domain D and Γ is any loop (closed contour)in D, then

$$\int_\Gamma f(z)\,dz = 0.$$

Theorem: In a simply connected domain, an analytic function has an antiderivative, its contour integrals are independent of path, and its loop integrals vanish.

Theorem (Cauchy's Integral Formula): Let Γ be a simple closed positively oriented contour. If $f(z)$ is analytic in some simply connected domain D containing Γ and z_0 is any point inside Γ, then

$$f(z_0)\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz.$$

Theorem: Let g be continuous on the contour Γ, and for each z not on Γ set

$$G(z)\,dz \equiv \int_\Gamma \frac{g(\xi)}{\xi - z}d\xi.$$

Then the function G is analytic, and its derivative is given by

$$G'(z)\,dz = \int_\Gamma \frac{g(\xi)}{(\xi - z)^2}d\xi$$

for all z not on Γ.

Theorem: If f is analytic in a domain D, then all its derivatives $f', f'', \ldots, f^{(n)}, \ldots$ exist and are analytic in D.

Theorem: If $f = u + iv$ is analytic in a domain D, then all partial derivatives of u and v exist and are continuous in D.

Theorem (Morera's Theorem): If $f(z)$ is continuous in a domain D and if

$$\int_\Gamma f(z)\,dz = 0$$

for every closed contour Γ in D, then $f(z)$ is analytic in D.

Theorem: If f is analytic inside and on the simple closed positively oriented contour Γ and if z is any point inside Γ, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi \qquad (n = 1, 2, 3, \ldots).$$

Another form of this equation

$$\int_\Gamma \frac{f(z)}{(z-z_0)^m}dz = \frac{2\pi i f^{(m-1)}(z_0)}{(m-1)!} \qquad (z_0 \text{ inside } \Gamma).$$

The Cauchy's residue theorem and method for calculating residues are quotes from http://math.furman.edu/ dcs/courses/math39/lectures/lecture-45.pdf.

Theorem (Cauchy's Residue Theorem): Suppose C is a positively oriented, simple closed contour. If f is analytic on and inside C except for finite number of singular points $z_1, z_2, \ldots, z_n$, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Method for calculating residues.
Let f be a function with a pole of order m at P . Then

$$\operatorname*{Res}_{z=P} f(z) = \frac{1}{(m-1)!}\left(\frac{\partial}{\partial z}\right)^{m-1}\left((z-P)^m f(z)\right)\Big|_{z=P}$$

Partial fraction decomposition.

$$\begin{aligned} \frac{N(z)}{D(z)} &= \frac{N(z)}{\cdots(z-P)^m\cdots} = \cdots + \sum_{\ell=1}^{m}\frac{A_\ell}{(z-P)^\ell} + \cdots \\ A_\ell &= \frac{1}{(\ell-1)!}\left(\frac{\partial}{\partial z}\right)^{\ell-1}\left((z-P)^m\frac{N(z)}{D(z)}\right)\Bigg|_{z=P} \end{aligned}$$

$$\begin{aligned} (a+b)^2 &= a^2+2ab+b^2 \\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3 \end{aligned}$$

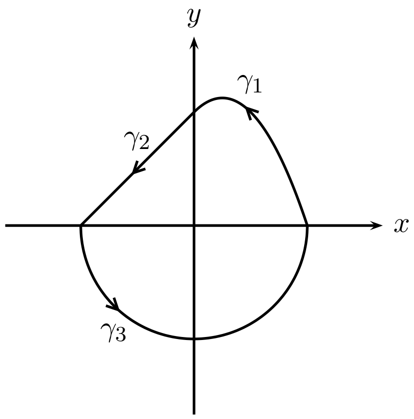
$$\begin{aligned} x &= \frac{z+\bar{z}}{2} \\ y &= \frac{z-\bar{z}}{2i} \end{aligned}$$

QUESTIONS

Q1) A simple closed contour; $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$, is given. The sequence of the directed smooth curves of this contour are given below.

$$\begin{aligned} \gamma_1 &: z_1(t) = 1 - t + i\left(3t - 2t^2\right), & 0 \le t \le 1 \\ \gamma_2 &: z_2(t) = -t + i(1-t), & 0 \le t \le 1 \\ \gamma_3 &: z_3(t) = \cos(\pi(t+1)) + i\sin(\pi(t+1)), & 0 \le t \le 1 \end{aligned}$$

Obtain a contour parametrization for Γ by employing the techniques of rescaling; rescale so that γ_1 is traced as t varies between 0 and $\frac{1}{3}$, γ_2 is traced for $\frac{1}{3} \le t \le \frac{2}{3}$, and γ_3 is traced for $\frac{2}{3} \le t \le 1$.



Q1

Q2) Compute

$$\left(e^{z_1} + e^{-z_1}\right) \cdot \left(e^{z_2} + e^{-z_2}\right) + \left(e^{z_1} - e^{-z_1}\right) \cdot \left(e^{z_2} - e^{-z_2}\right)$$

Using the result obtained find the trigonometric identity for $\cosh(z_1 + z_2)$.
$(a+b)(c+d) + (a-b)(c-d) = 2ac + 2bd$.

Q3) Compute

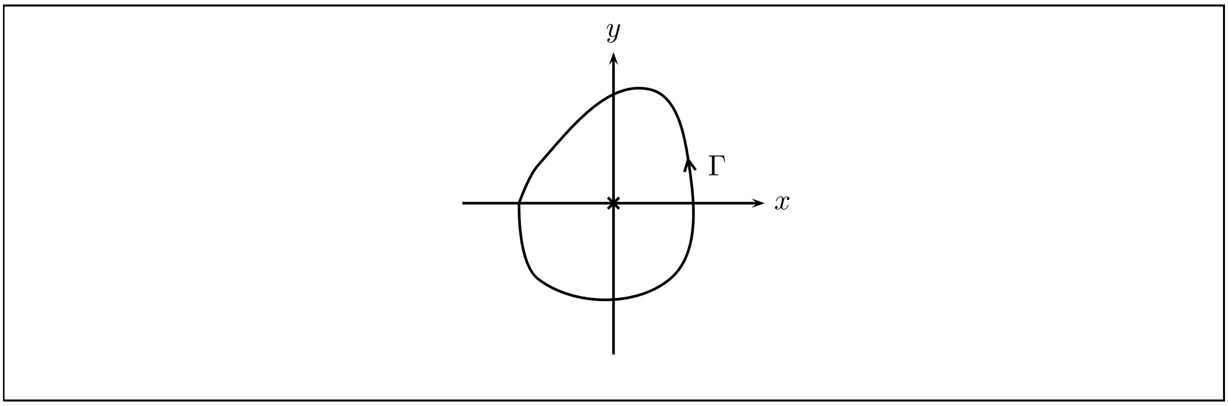
$$\left(e^{z_1} + e^{-z_1}\right) \cdot \left(e^{z_2} - e^{-z_2}\right) + \left(e^{z_1} - e^{-z_1}\right) \cdot \left(e^{z_2} + e^{-z_2}\right)$$

Using the result obtained find the trigonometric identity for $\sinh(z_1 - z_2)$.
$(a+b)(c-d) + (a-b)(c+d) = 2ac - 2bd$.

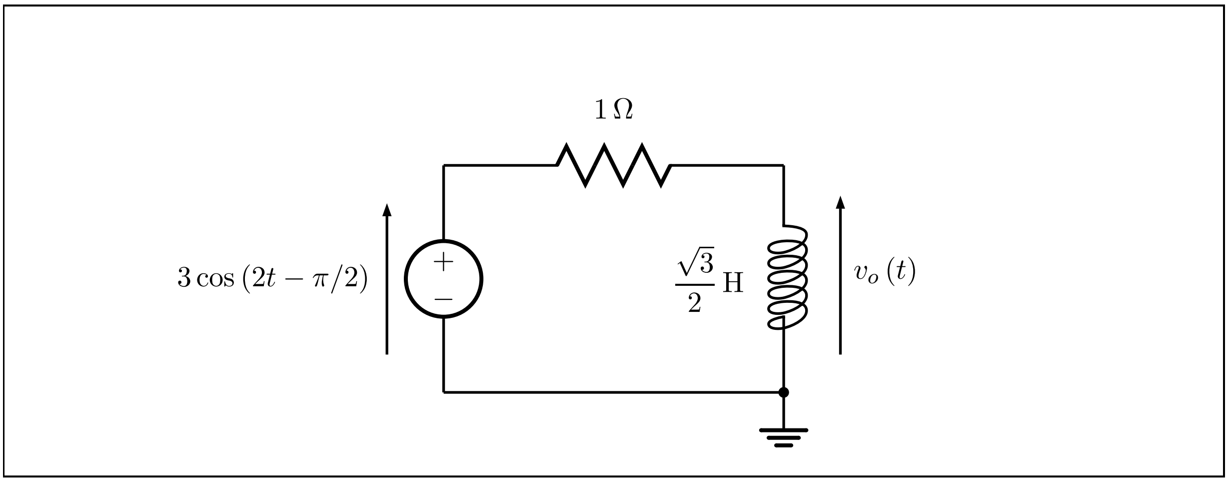
Q4) Evaluate the following contour integral.

$$\int_\Gamma \frac{-2z^2 + z + 3}{z^3}\, dz$$

Q5) In the following circuit AC source; $3\cos(2t - \pi/2)$ Volt is connected for a long time and the responses to this source are at steady state. obtain the steady state voltage $v_o(t)$ on the



Q4



Q5

inductor.

Q6) A complex function $u(x, y) = 2xy$ is given. Is it is harmonic? If yes, find harmonic conjugate of this function.

Q7) A complex function $u(x, y) = \frac{x}{x^2 + y^2}$ is given. Is it is harmonic? If yes, find harmonic conjugate of this function.

Q8) The complex logarithm function is defined as in the following.

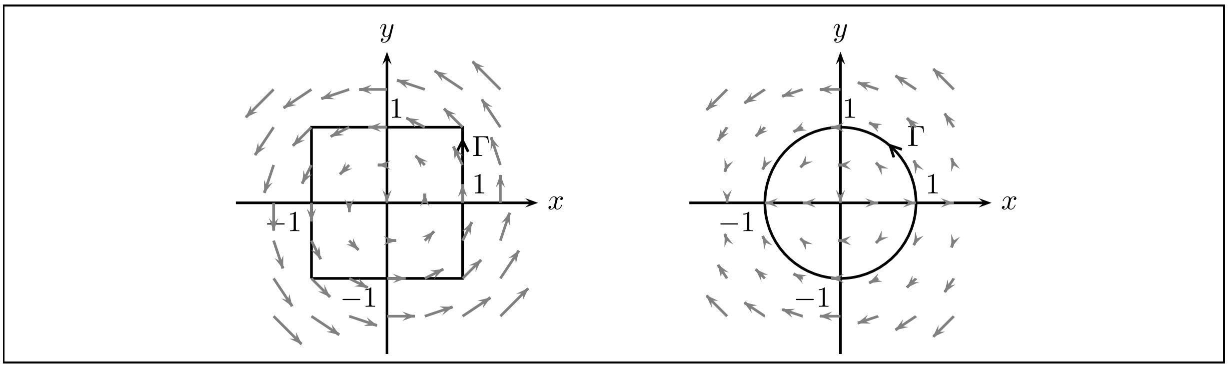
$$\begin{aligned} z &= re^{i\theta} \\ \log(z) &= \mathrm{Log}(r) + \theta + 2\pi k, \qquad k \in \mathbb{Z} \end{aligned}$$

A branch of the logarithm is

$$\mathcal{L}_0(z) = \mathrm{Log}(r) + \theta, \qquad \text{for } 0 < \theta \leq 2\pi.$$

What is the domain that $\mathcal{L}_0(z)$ is analytic. Find $\mathcal{L}_0\left(1 + i\,0^+\right)$, and $\mathcal{L}_0\left(1 + i\,0^-\right)$. Obtain $\frac{d}{dz}\mathcal{L}_0(z)$.

Q9) Two vectors fields; $\mathbf{V}(x,y) = (-y,\, x)$, and $\mathbf{V}(x,y) = \left(-y^2,\, xy\right)$ are given. For each of the vector field and the following contours test Green's theorem.



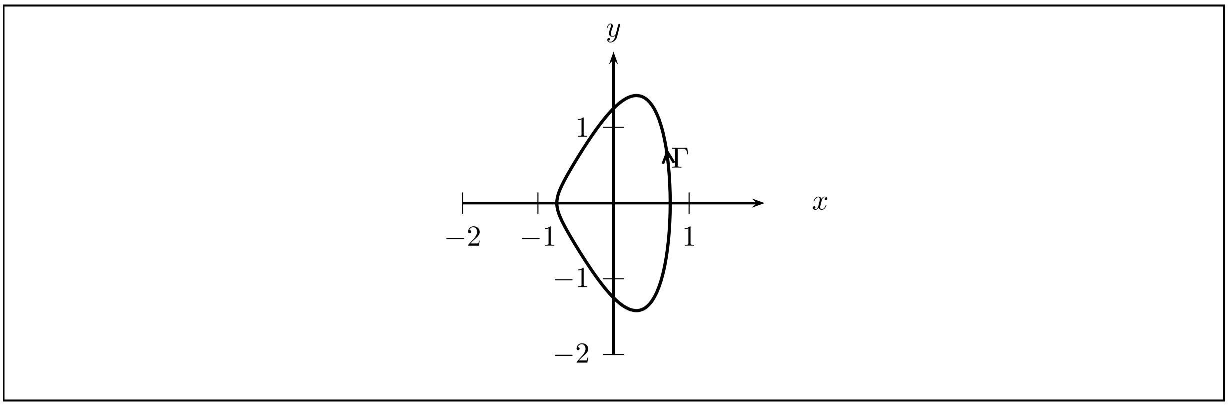
Q9

Q10) Find the singularities and the residues of the following complex function.

$$f(z) = (2+i)\,z^2 - 3iz + 1 - 2i + \frac{1-i}{z-2i} + \frac{1+i}{z+2i} - \frac{3}{(z+2)^2} + \frac{2}{z+2}$$

Q11) Compute the following integral along the simple closed contour Γ traversed once counter clockwise direction.

$$\int_\Gamma \frac{1-\cos(z)}{z\,(z+i)^2}\,dz$$



Q11

Q12) Evaluate the following integral along the simple closed contour Γ traversed once counter clockwise direction.

$$\int_\Gamma \frac{e^z \sin^2(z)}{z\,(z^2+4)}\,dz$$

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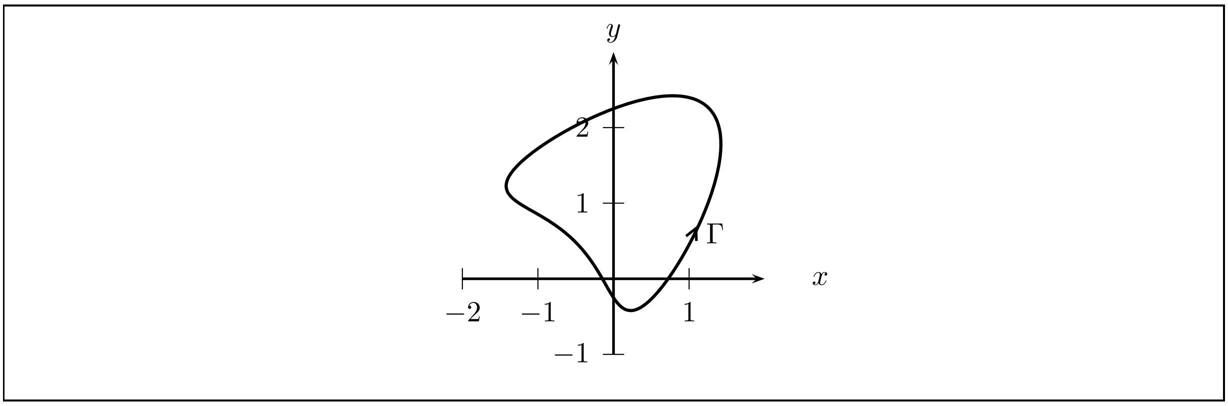

Q12