Instructions Answer all questions. Give your answers clearly. Do not skip intermediate steps even they are very easy. Complete answers with no error, including expression and notation errors receive full mark.

Calculators are not allowed. Cell phones have to be switched off and be kept in your pocked or in your bag.

Time 90 minutes.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
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| TOTAL |  |  |  |  |  |  |  |  |  |
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## QUESTIONS

Q1) (14p) Evaluate the following mathematical operations. Write your results in Cartesian form.
a) $6 e^{i \pi / 3}+\frac{4}{1+i \sqrt{3}}$
b) $2(-1)^{1 / 3}$

Q2) (9p) A complex function is given as $f(z)=z^{2}+2 \bar{z}$. Replace complex variable $z$ with its Cartesian form $z=x+i y$ in the function and obtain $w(x, y)=f(x+i y)$. What are the real and imaginary parts of $w(x, y)$.

Q3) (9p) A complex function $w(x, y)=x^{2}-y^{2}+2 x+\mathfrak{i}(2 x y-2 y)$ is given. Here $x$ and $y$ are real and imaginary parts of complex variable $z=x+i y$. Check analyticity of the function with Cauchy-Riemann equations. Is it analytic ?

Q4) (15p) Derive the description of the inverse cosine function $w=\cos ^{-1}(z)$ in terms of $\log$ function.
(Hint: Write $\cos (w)$ in terms of complex exponentials.)

Q5) (10p) Evaluate the following integral.
$\int_{0}^{1} \frac{20 t}{\left(t^{2}+1-i\right)^{2}} d t$
(Hint: change the term in the parenthesis with a new variable.)

Q6) (6p) Decide if the following domains are simply connected or not. Write your reasoning ?
a)

b)


Q7) (4p) Which of the following loops are positively oriented (counter-clockwise direction) ?
a)

b)


Q8) (15p) Evaluate the following line integral. The line is $\Gamma: z(t)=\log (2)+\mathfrak{i}(t \pi / 3-\pi / 6)$, $0 \leq \mathrm{t} \leq 1$. When it is necessary, take $\log (2) \approx 0.7$ and $\sqrt{3} \pi \approx 5.4$.
$\int_{\Gamma}(z+1) e^{z} d z=?$
(Hint: $\frac{d}{d z} z e^{z}=? \quad z-\bar{z}=?($ or $a+i b-(a-i b)=?)$

Q9) (9p) Evaluate the following integral. $\Gamma$ is a loop with any point $z$ on it satisfy the inequality $|z|<3 / 2$. The loop traverses in the counter-clockwise direction.
$\int_{\Gamma} \frac{e^{z}}{z-1} d z=?$
(Hint : Remember Cauchy's integral formula. $\oint_{C} \frac{f(z)}{z-z_{0}} d z=$ ?)
Q10) (9p) Compute line integral, $\int_{\Gamma} f(z) d z$, of the following function
$f(z)=\frac{10-2 z}{z^{2}-2 z-3}=\frac{1}{z-3}-\frac{3}{z+1}$

Here $\Gamma$ is a loop with any point $z$ on it satisfy the inequality $|z-1|<3$. The loop traverses in the counter-clockwise direction.
(Hint: Remember the integral; $\oint_{C}(z-a)^{n} d z=$ ?)

## ANSWERS

A1)
a)
$6 e^{i \pi / 3}=6 \cos (\pi / 3)+i 6 \sin (\pi / 3)=3+i 3 \sqrt{3}$
$\frac{4}{1+i \sqrt{3}}=\frac{4(1-i \sqrt{3})}{(1+i \sqrt{3})(1-\mathfrak{i} \sqrt{3})}=\frac{4(1-\mathfrak{i} \sqrt{3})}{4}=1-\mathfrak{i} \sqrt{3}$
$6 e^{i \pi / 3}+\frac{4}{1+i \sqrt{3}}=3+i 3 \sqrt{3}+1-i \sqrt{3}=4+i 2 \sqrt{3}$
b)
$-1 \quad=e^{\pi+2 \pi k}, \quad k \in \mathbb{Z}$
$(-1)^{1 / 3}=e^{\pi / 3+2 \pi k / 3}, \quad \mathrm{k}=0,1,2$
$2(-1)^{1 / 3}=2 e^{\pi / 3+2 \pi k / 3}, \quad \mathrm{k}=0,1,2$
$(-1)^{1 / 3}$ has three roots and therefore $2(-1)^{1 / 3}$ has three answers.
$k=0: 2 e^{\pi / 3+2 \pi k / 3}=2 e^{\pi / 3}=2 \cos (\pi / 3)+i 2 \sin (\pi / 3)=1+i \sqrt{3}$
$k=1: 2 e^{\pi / 3+2 \pi k / 3}=2 e^{\pi}=2 \cos (\pi)+i 2 \sin (\pi)=-2$
$k=2: 2 e^{\pi / 3+2 \pi k / 3}=2 e^{5 \pi / 3}=2 \cos (5 \pi / 3)+i 2 \sin (5 \pi / 3)=1-i \sqrt{3}$

A2)

$$
\begin{aligned}
w(x, y) & =(x+i y)^{2}+2 \overline{x+i y} \\
& =x^{2}+i 2 x y+(i y)^{2}+2(x-i y) \\
& =x^{2}+i 2 x y-y^{2}+2 x-i 2 y
\end{aligned}=x^{2}-y^{2}+2 x+i(2 x y-2 y)
$$

$\operatorname{Re}[w(x, y)]=x^{2}-y^{2}+2 x$
$\operatorname{Im}[w(x, y)]=2 x y-2 y$

A3)
$u(x, y)=\operatorname{Re}[w(x, y)]=x^{2}-y^{2}+2 x$
$v(x, y)=\operatorname{Im}[w(x, y)]=2 x y-2 y$
$\begin{aligned} \frac{\partial}{\partial x} u(x, y) & =2 x+2 & \frac{\partial}{\partial y} u(x, y) & =-2 x \\ \frac{\partial}{\partial x} v(x, y) & =2 y & \frac{\partial}{\partial y} v(x, y) & =2 x-2\end{aligned}$
We check if the the Cauchy-Riemann equations are satisfied. For the given function,

$$
\begin{aligned}
\frac{\partial}{\partial x} u(x, y) & \neq \frac{\partial}{\partial y} v(x, y) \\
\frac{\partial}{\partial y} u(x, y) & =-\frac{\partial}{\partial x} v(x, y)
\end{aligned}
$$

The equality is fulfilled in the second expression but not in the first expression for any $x$ and $y$, therefore the function is nowhere analytic.

You should immediately realize that the function given in this question is the same with the one given in the second equation. This function contains $\bar{z}$ in its definition. $\bar{z}$ is nowhere analytic so $f(z)$. Recall the differentiation of $\bar{z}$

$$
\begin{array}{ll}
\frac{\mathrm{d} \bar{z}}{\mathrm{~d} z}=1 & \text { when } \mathrm{d} z=\mathrm{d} x \text { (when we approach to } z \text { along real axis) } \\
\frac{\mathrm{d} \bar{z}}{\mathrm{~d} z}=-1 & \text { when } \mathrm{d} z=\mathrm{idy} \text { (when we approach to } z \text { along imaginary axis) }
\end{array}
$$

The derivative should be independent of any direction we approach. Consequently $\bar{z}$ is nowhere analytic since there isn't any domain it is analytic in the complex plane.

A4) We can write $w=\cos ^{-1}(z)$ as $\cos (w)=z$ to escape from the inverse function. Using Euler's formula $\cos (w)$ is replaced with exponentials.
$\frac{e^{i w}+e^{-i w}}{2}=z$

Continue manipulating this equation

$$
\begin{array}{ll}
e^{i w}+e^{-i w} & =2 z \\
e^{i 2 w}+1 & =2 z e^{i w} \\
e^{i 2 w}-2 z e^{i w}+1 & =0
\end{array}
$$

We need to compute roots of the second order equation
$\left(e^{i w}\right)^{2}-2 z e^{i w}+1=0$

$$
\begin{aligned}
& \Delta=\sqrt{(-z)^{2}-1 \cdot 1}=\sqrt{z^{2}-1} \\
& e^{i w}=\frac{-(-z)+\Delta}{1}=z+\sqrt{z^{2}-1}
\end{aligned}
$$

Then we take logarithm of the both sides of the last equation to extract variable $w$

$$
\begin{aligned}
\mathfrak{i} w & =\log \left(z+\sqrt{z^{2}-1}\right) \\
w & =-\mathfrak{i} \log \left(z+\sqrt{z^{2}-1}\right) \quad \text { This is the answer }
\end{aligned}
$$

A5) Let us change $t^{2}+1-i$ with a new variable $s$.

$$
\begin{aligned}
& \mathrm{t}^{2}+1-\mathrm{i}=\mathrm{s} \quad 2 \mathrm{tdt}=\mathrm{d} s \\
& \mathrm{t}=0 \Rightarrow \mathrm{~s}=1-\mathrm{i} \\
& \mathrm{t}=1 \Rightarrow \mathrm{~s}=2-\mathrm{i}
\end{aligned}
$$

Re-write the integral with the new variable and compute the integral

$$
\begin{aligned}
\int_{0}^{1} \frac{20 t}{\left(t^{2}+1-i\right)^{2}} d t & =\int_{1-i}^{2-i} \frac{10}{s^{2}} d s \\
& =-\left.\frac{10}{s}\right|_{1-i} ^{2-i} \frac{10}{s^{2}} d s \\
& =\frac{-10+10 i+20-10 i}{1-3 i}
\end{aligned} \begin{aligned}
& 2-i \\
& 1-i \\
& 1-i \\
& \\
&
\end{aligned}
$$

A6)
a) Any loop inside the domain stretches a single point in the domain. Equivalently, the interior of any loop in the domain is completely inside the domain. Therefore, this domain is simply connected.
b) This loop do not meet the conditions mentioned above. It is not simply connected.

A7)
a) The direction of the loop is counter-clockwise (opposite direction of the clockwise direction). Equivalently, while the loop direction is upward the interior of the loop is at the left side. Therefore direction of the loop is positively oriented.
b) The direction of the loop is clockwise. Equivalently, while the loop direction is upward the interior of the loop is at the right side. Therefore direction of the loop is not positively oriented.

A8) The functions $z+1$ and $e^{z}$ are entire functions. So, $(z+1) e^{z}$ is entire. Anti-derivative of the integrand $f(z)=(z+1) e^{z}$ is $F(z)=z e^{z}$. Then,

$$
\int_{\Gamma}(z+1) e^{z} \mathrm{~d} z=\left.z e^{z}\right|_{z(0)} ^{z(1)}=z(1) e^{z(1)}-z(0) e^{z(0)}
$$

Here,

$$
\begin{aligned}
& z(0)=\log (2)+\mathfrak{i}(0 \cdot \pi / 3-\pi / 6)=\log (2)-\mathfrak{i} \pi / 6 \\
& z(1)=\log (2)+\mathfrak{i}(1 \cdot \pi / 3-\pi / 6)=\log (2)+\mathfrak{i} \pi / 6
\end{aligned}
$$

$$
z(0)=\overline{z(1)}
$$

$$
z(0) e^{z(0)}=\overline{z(1)} e^{\overline{z(1)}}=\overline{z(1) e^{z(1)}}
$$

Therefore,

$$
z(1) e^{z(1)}-z(0) e^{z(0)}=i 2 \operatorname{Im}\left[z(1) e^{z(1)}\right]
$$

$$
\begin{aligned}
z(1) e^{z(1)} & =[\log (2)+\mathfrak{i} \pi / 6] e^{\log (2)+i \pi / 6} \\
& =\log (2) e^{\log (2)+\mathfrak{i} \pi / 6}+\mathfrak{i} \frac{\pi}{6} \mathrm{~L}^{\log (2)+\mathfrak{i} \pi / 6} \\
& =2 \log (2) e^{\mathfrak{i} \pi / 6}+\mathfrak{i} \frac{\pi}{3} e^{\mathfrak{i} \pi / 6}
\end{aligned}
$$

We need to compute

$$
\begin{aligned}
\operatorname{Im}\left[z(1) e^{z(1)}\right] & =\operatorname{Im}\left[2 \log (2) e^{i \pi / 6}+i \frac{\pi}{3} e^{i \pi / 6}\right] \\
& =\operatorname{Im}\left[2 \log (2) e^{i \pi / 6}\right]+\operatorname{Im}\left[i \frac{\pi}{3} e^{i \pi / 6}\right] \\
& =2 \log (2) \sin (\pi / 6)+\frac{\pi}{3} \cos (\pi / 6) \\
& =\log (2)+\frac{\pi \sqrt{3}}{6}
\end{aligned}
$$

It follows

$$
\begin{aligned}
& \int_{\Gamma}(z+1) e^{z} \mathrm{~d} z=\mathfrak{i 2 \operatorname { I m } [ z ( 1 ) e ^ { z ( 1 ) } ]} \\
&=\mathfrak{i}\left(2 \log (2)+\frac{\pi \sqrt{3}}{3}\right) \\
&=\mathfrak{i}(1.4+1.8)=\mathfrak{i 3 . 2}
\end{aligned}
$$

A9) We will employ Cauchy's integral formula.
$\oint_{C} \frac{f(z)}{z-z_{0}} d z=2 \pi i f\left(z_{0}\right)$

The direction of loop $C$ is counter-clockwise and encloses the point $z_{0} . f(z)$ is analytic on and at the interior of the loop.
$e^{z}$ is entire and direction of the loop is counter-clockwise and the loop $\Gamma$ encloses $z=1$. Therefore,
$\int_{\Gamma} \frac{e^{z}}{z-1} \mathrm{~d} z=2 \pi i e^{1}=2 \pi i e$

A10) $z=3$ and $z=-1(f(3)= \pm \infty, f(-1)= \pm \infty)$ are poles of $f(z)$. The loop $\Gamma$ encloses the poles and its direction is positively oriented. Then,
$\int_{\Gamma} f(z) d z=\oint_{C_{1}} f(z) d z+\oint_{C_{2}} f(z) d z$
with

$$
C_{1}: r_{1}(t)=3+\epsilon_{1} e^{i 2 \pi t} \quad \text { and } \quad r_{2}(t)=-1+\epsilon_{2} e^{i 2 \pi t} \quad 0 \leq t \leq 1
$$

$$
\begin{aligned}
\oint_{\mathrm{C}_{1}} \mathrm{f}(z) \mathrm{d} z & =\oint_{\mathrm{C}_{1}} \frac{1}{z-3} \mathrm{~d} z-3 \oint_{\mathrm{C}_{1}} \frac{1}{z+1} \mathrm{~d} z \\
& =2 \pi i-3 \cdot 0 \quad=2 \pi i
\end{aligned}
$$

$$
\oint_{\mathrm{C}_{2}} \mathrm{f}(z) \mathrm{d} z=\oint_{\mathrm{C}_{2}} \frac{1}{z-3} \mathrm{~d} z-3 \oint_{\mathrm{C}_{2}} \frac{1}{z+1} \mathrm{~d} z
$$

$$
=0-3 \cdot 2 \pi i \quad=-6 \pi i
$$

Concequently,
$\int_{\Gamma} f(z) d z=2 \pi i-6 \pi i=-4 \pi i$

