Instructions Answer all questions. Give your answers clearly. Do not skip intermediate steps even they are very easy. Complete answers with no error, including expression and notation errors receive full mark.
Calculator and cell phone are not allowed in the exam.
Time 90 minutes.
Good Luck.

1	2	3	4	5	TOTAL

QUESTIONS

Q1) (27 p) Evaluate the following expressions. Obtain possible simplest form of the results in cartesian coordinates.

a) $\dfrac{2+3i}{-1+4i}+\dfrac{7}{17}-\dfrac{23}{17}i$, b) $2e^{i\pi/12}\cdot\left(3e^{i\pi/4}-i\left(e^{-i\pi/8}\right)^2\right)$, c) $\left(\sqrt{2}+i\sqrt{2}\right)^2(-i)^{1/3}$

Q2) (27 p) Evaluate the principle value (i.e., the value given by the principle branch) of the following expressions.

a) $\sin(\pi/6+i\mathrm{Log}(2))$, b) $(-1)^{\left(1+i\mathrm{Log}(3)/\pi\right)}$, c) $\tanh(\mathrm{Log}(2)+i\pi/6)$.

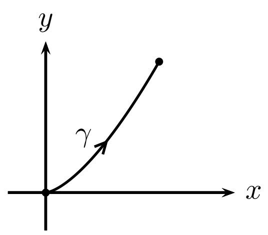
Q3) (15 p) Find a description of $\sinh^{-1}(z)$ in terms of logarithms.

Q4) (20 p) A harmonic function;

$$u(x,y)=\frac{1}{2}\mathrm{Log}\left(x^2+y^2\right)$$

is given. Show that the function is harmonic. Compute harmonic conjugate; $v(x,y)$ of this function.

Q5) (11 p) A smooth arc; $\gamma\,:\,z(t)=\dfrac{1}{2}t^2+i\dfrac{1}{3}t^3$, $0\le t\le\sqrt{3}$, is given. Compute length; $l(\gamma)$, of the smooth arc.



Q5

$$\frac{d}{dz}\left(1+z^2\right)\sqrt{1+z^2} = 3z\sqrt{1+z^2}$$

De Moivre's formula

$$[\, r\cos(\theta) + i\, r\sin(\theta)\,]^n = r^n\cos(n\theta) + i\, r^n \sin(n\theta)$$

Complex exponential function

$$e^z = \cos(z) + i\sin(z)$$

$$l(\gamma) = \int_\gamma \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2}\, dt$$

$$\begin{aligned} \log(z) &= \mathrm{Log}(|z|) + i\,\mathrm{Arg}(z) + i\,2\pi k, \qquad k \in \mathbb{Z} \\ z^\alpha &= e^{\alpha \log(z)} \end{aligned}$$

$$\begin{aligned} \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ \sinh(z) &= \frac{e^{z} - e^{-z}}{2} \\ \tanh(z) &= \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} \end{aligned}$$

$$\begin{aligned} \frac{d}{dz}\tan^{-1}(z) &= \frac{1}{1+z^2} \\ \frac{d}{dz}\tan^{-1}(z/a) &= \frac{a}{a^2+z^2} \\ \frac{d}{dz}\tan^{-1}(a/z) &= \frac{-a}{a^2+z^2} \end{aligned}$$

ANSWERS

A1) a)

$$\begin{aligned}
\frac{2+3i}{-1+4i} &= \frac{(2+3i)(-1-4i)}{(-1-4i)(-1+4i)} \\
&= \frac{2\cdot(-1)+2\cdot(-4i)+3i\cdot(-1)+3i\cdot(-4i)}{(-1)^2-(4i)^2} \\
&= \frac{10-11i}{17} = \frac{10}{17}-\frac{11}{17}i
\end{aligned}$$

$$\begin{aligned}
\frac{2+3i}{-1+4i}+\frac{7}{17}-\frac{23}{17}i &= \frac{10}{17}-\frac{11}{17}i+\frac{7}{17}-\frac{23}{17}i \\
&= \boxed{1-2i}
\end{aligned}$$

b)

$$\begin{aligned}
2e^{i\pi/12}\cdot\left(3e^{i\pi/4}-i\left(e^{-i\pi/8}\right)^2\right) &= 2e^{i\pi/12}\cdot\left(3e^{i\pi/4}-ie^{-i\pi/4}\right) \\
&= 6e^{i(\pi/12+\pi/4)}-i2e^{i(\pi/12-\pi/4)} \\
&= 6e^{i\pi/3}-i2e^{-i\pi/6} \\
&= 6e^{i\pi/3}-2e^{i\pi/2}e^{-i\pi/6} \\
&= 6e^{i\pi/3}-2e^{i(\pi/2-\pi/6)} \\
&= 6e^{i\pi/3}-2e^{i\pi/3} \\
&= 4e^{i\pi/3} \\
&= 4\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right) \\
&= 2+i2\sqrt{3}
\end{aligned}$$

c)

$$
\begin{aligned}
-i &= e^{-i\pi/2} \\
|-i| &= 1 \\
\arg(-i) &= -\frac{\pi}{2} + 2\pi k \\
\left|(-i)^{1/3}\right| &= |-i|^{1/3} && = 1 \\
\arg\left((-i)^{1/3}\right) &= \frac{1}{3}\arg(-i) && = -\frac{\pi}{6} + \frac{2\pi}{3}k \\
(-i)^{1/3} &= \cos\left(-\frac{\pi}{6} + \frac{2\pi}{3}k\right) + i\sin\left(-\frac{\pi}{6} + \frac{2\pi}{3}k\right) && k = 0, 1, 2
\end{aligned}
$$

by De Moivre's formula

Roots

$k = 0;$ $\quad \frac{\sqrt{3}}{2} - i\frac{1}{2}$

$k = 1;$ $\quad i$

$k = 2;$ $\quad -\frac{\sqrt{3}}{2} - i\frac{1}{2}$

$$\left(\sqrt{2} + i\sqrt{2}\right)^2 = 4i$$

The expression $\left(\sqrt{2} + i\sqrt{2}\right)^2 (-i)^{1/3}$ has three values.

$$
\begin{aligned}
k = 0; \quad & 4i \cdot \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) &&= \boxed{2 + i2\sqrt{3}} \\
k = 1; \quad & 4i \cdot i &&= \boxed{-4} \\
k = 2; \quad & 4i \cdot \left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) &&= \boxed{2 - i2\sqrt{3}}
\end{aligned}
$$

A2) a)

$$
\begin{aligned}
\sin(\pi/6 + i\mathrm{Log}(2)) &= \frac{e^{i\left(\pi/6+i\mathrm{Log}(2)\right)} - e^{-i\left(\pi/6+i\mathrm{Log}(2)\right)}}{2i} \\
&= \frac{e^{\left(i\pi/6-\mathrm{Log}(2)\right)} - e^{\left(-i\pi/6+\mathrm{Log}(2)\right)}}{2i} \\
&= \frac{e^{-\mathrm{Log}(2)}e^{i\pi/6} - e^{\mathrm{Log}(2)}e^{-i\pi/6}}{2i} \\
&= \frac{(1/2)\,e^{i\pi/6} - 2e^{-i\pi/6}}{2i} \\
&= -i\,\frac{1}{4}e^{i\pi/6} + i\,e^{-i\pi/6} \\
&= -i\,\frac{1}{4}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) + i\,2\left(\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right) \\
&= -i\,\frac{\sqrt{3}}{8} + \frac{1}{8} + i\,\sqrt{3} + 1 \\
&= \boxed{i\,\frac{7\sqrt{3}}{8} + \frac{9}{8}}
\end{aligned}
$$

b) $(-1)^{\left(1+i\mathrm{Log}(3)/\pi\right)}$

$$
\begin{aligned}
|-1| &= 1 \\
\arg(-1) &= \pi + 2\pi k \\
-1 &= e^{i(\pi+2\pi k)} \\
(-1)^{\left(1+i\mathrm{Log}(3)/\pi\right)} &= -(-1)^{i\mathrm{Log}(3)/\pi} \\
&= -e^{i(\pi+2\pi k)\cdot i\mathrm{Log}(3)/\pi} \\
&= -e^{-\mathrm{Log}(3)\cdot(1+2k)} \\
&= -\left(\frac{1}{3}\right)^{1+2k} \\
\text{the principle value } (k=0) &= \boxed{-\frac{1}{3}}
\end{aligned}
$$