Instructions Answer all questions. Give your answers clearly. Do not skip intermediate steps even they are very easy. Complete answers with no error, including expression and notation errors receive full mark.
Calculator and cell phone are not allowed in the exam.
Time 75 minutes.
Good Luck.

1	2	3	4	5	TOTAL

QUESTIONS

Q1) Compute the following limits. Note that a complex number $z = re^{i\theta}$ is zero if $r = 0$ and is infinite if $r = \infty$.

a) $\lim\limits_{t\to\infty} \dfrac{e^{(-1+2i)t}}{1+e^{it}}$, b) $\lim\limits_{z\to 1+i} \dfrac{z^2 - 2iz + 1 - i}{z - 2 + i}$

Q2) A complex function in polar coordinates is given as in the following.

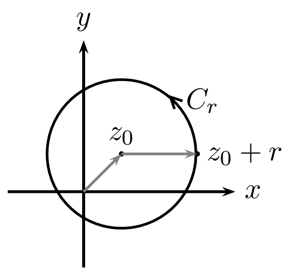
$$f(z) = 1 + 2r^2 \cos(2\theta) - 3r^3 \cos(3\theta) + i\left(2r^2 \sin(2\theta) - 3r^3 \sin(3\theta)\right)$$

Obtain the representation of $f(z)$ in terms of complex variable z. Find real part; $u(x, y)$, and imaginary part; $v(x, y)$, in cartesian coordinates.
Employ De Moivre's formula to find the solution.

Q3) Compute $\dfrac{d}{dz} \sinh^{-1}(z)$.
Use the definition of $\sinh^{-1}(z)$ in terms of logarithms; $\sinh^{-1}(z) = \log\left(z + \sqrt{z^2 + 1}\right)$.

Q4) Evaluate the following line integral along the closed curve; $C_r : z(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi$. Here, $n \in \mathbb{Z}$.
Use parametric integration method to find the integral.

$$\oint_{C_r} (z - z_0)^n \, dz$$



Q4

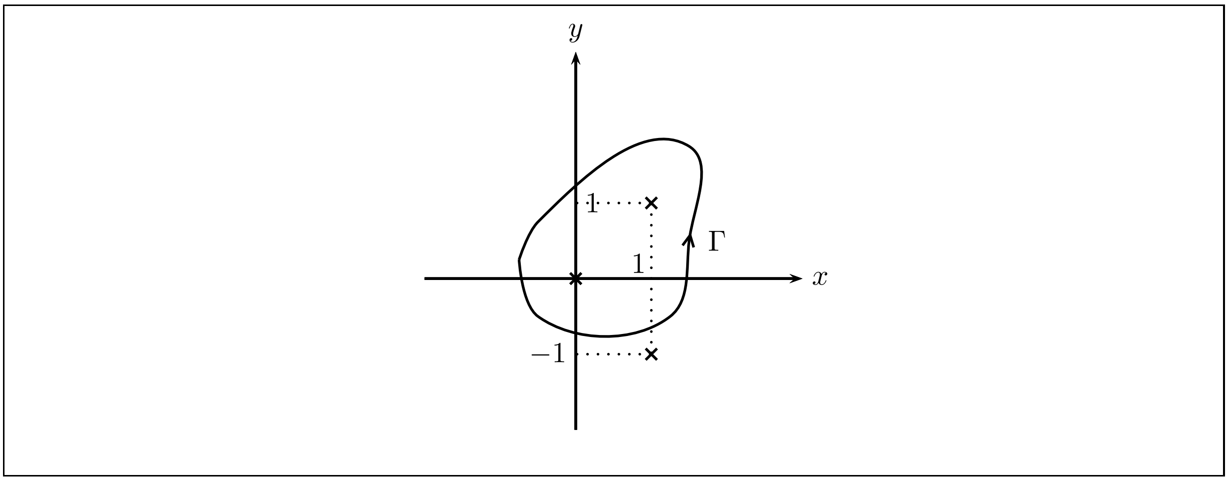
Q5) Find the following contour integral.

$$\int_{\Gamma} \frac{-2z^2+8z-4}{z^3-2z^2+2z}\, dz$$

First obtain partial fraction expansion of the integrand. The roots of the denominator $z^3 - 2z^2 + 2z$ are 0, $1-i$, and $1+i$ respectively. Therefore the partial fraction expansion of the integrand is as follows.

$$\frac{-2z^2+8z-4}{z^3-2z^2+2z} = \frac{A}{z} + \frac{B}{z-1+i} + \frac{C}{z-1-i}$$

If z_0 is a root of the denominator of a rational polynomial $f(z)$, a partial fraction of $f(z)$ is $\frac{K}{z-z_0}$ and the constant K can be found by $K = \lim_{z \to z_0} (z-z_0) f(z)$.



Q5

ANSWERS

A1) a)

$$\lim_{t\to\infty} \frac{e^{(-1+2i)t}}{1+e^{it}} = \lim_{t\to\infty} \frac{e^{-t}e^{i2t}}{1+e^{it}}$$

$$\lim_{t\to\infty} e^{-t} = e^{-\infty} = 0$$

$$\lim_{t\to\infty} e^{i2t} = e^{i\cdot\infty} = z_1, \qquad \lim_{t\to\infty} e^{it} = e^{i\cdot\infty} = z_2$$

z_1 and z_2 are complex numbers with unit absolute value.

$$\lim_{t\to\infty} \frac{e^{-t}e^{i2t}}{1+e^{it}} = \frac{0\cdot z_1}{1+z_2} = \boxed{0}$$

b)

$$\begin{aligned}
\lim_{z\to 1+i} \frac{z^2-2iz+1-i}{z-2+i} &= \frac{(1+i)^2-2i\,(1+i)+1-i}{1+i-2+i} \\
&= \frac{2i-2i+2+1-i}{-1+2i} \\
&= \frac{3-i}{-1+2i} \\
&= \frac{(3-i)\,(-1-2i)}{(-1+2i)\,(-1-2i)} \\
&= \frac{-5-5i}{5} \\
&= \boxed{-1-i}
\end{aligned}$$

A2)

$$\begin{aligned}
f(z) &= 1+2r^2\cos(2\theta)-3r^3\cos(3\theta)+i\left(2r^2\sin(2\theta)-3r^3\sin(3\theta)\right) \\
&= 1+2\left[\,r^2\cos(2\theta)+i\,r^2\sin(2\theta)\,\right]-3\left[\,r^3\cos(3\theta)+i\,r^3\sin(3\theta)\,\right] \\
&= 1+2\,[\,r\cos(\theta)+i\,r\sin(\theta)\,]^2-3\,[\,r\cos(\theta)+i\,r\sin(\theta)\,]^3
\end{aligned}$$

$$\boxed{f(z) = 1+2z^2-3z^3}$$

$$
\begin{aligned}
f(z) &= 1 + 2z^2 - 3z^3 \\
&= 1 + 2\,[\,x + iy]^2 - 3\,[\,x + iy\,]^3 \\
&= 1 + 2\left[x^2 + 2iyx + (iy)^2\right] - 3\left[x^3 + 3x\,(iy)^2 + 3x^2 iy + (i\,y)^3\right] \\
&= 1 + 2\,[\,x^2 - y^2 + i2yx] - 3\,[\,x^3 - 3xy^2 + i3x^2y - iy^3\,] \\
&= 1 + 2x^2 - 2y^2 + 9xy^2 - 3x^3 + i\,[\,4yx - 9x^2y + 3y^3\,]
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
u(x,y) &= 1 + 2x^2 - 2y^2 + 9xy^2 - 3x^3 \\
v(x,y) &= 4yx - 9x^2y + 3y^3
\end{aligned}
}
$$

A3)

$$
\begin{aligned}
\frac{d}{dz}\sinh^{-1}(z) &= \frac{d}{dz}\log\left(z + \sqrt{z^2+1}\right) \\
&= \frac{1}{z + \sqrt{z^2+1}}\,\frac{d}{dz}\left(z + \sqrt{z^2+1}\right) \\
&= \frac{1}{z + \sqrt{z^2+1}}\left(1 + \frac{1}{2}\,\left(z^2+1\right)^{-1/2}\,2z\right) \\
&= \frac{1}{z + \sqrt{z^2+1}}\left(1 + z\,\left(z^2+1\right)^{-1/2}\right) \\
&= \frac{1}{z + \sqrt{z^2+1}}\,\frac{z + \sqrt{z^2+1}}{\sqrt{z^2+1}}
\end{aligned}
$$

$$
\boxed{\frac{d}{dz}\sinh^{-1}(z) = \frac{1}{\sqrt{z^2+1}}}
$$

A4)

$$\Gamma\,:\,z\,=\,z(t)\,,\quad a \le t \le b$$

$$\int_\Gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$
\begin{aligned}
z(t) &= z_0 + re^{it}, \qquad 0 \le t \le 2\pi \\
z'(z) &= ire^{it} \\
f(z) &= (z - z_0)^n \\
f(z(t)) &= \left(z_0 + re^{it} - z_0\right)^n = r^n e^{int} \\
\oint_{C_r} (z - z_0)^n \, dz &= \int_0^{2\pi} r^n e^{int} \cdot ire^{it} \, dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt
\end{aligned}
$$

For $n \neq -1$

$$
\begin{aligned}
ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt &= \frac{r^{n+1}}{n+1} e^{i(n+1)t} \Big|_0^{2\pi} = \frac{r^{n+1}}{n+1} \left[e^{i(n+1)2\pi} - e^{i(n+1)\cdot 0} \right] \\
&= \frac{r^{n+1}}{n+1} [1 - 1] = 0
\end{aligned}
$$

Note that for $n = -1$, $\dfrac{r^{n+1}}{n+1} e^{i(n+1)t}$ is undefined.

For $n = -1$

$$
\begin{aligned}
ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt &= ir^0 \int_0^{2\pi} e^{i\cdot 0\cdot t} \, dt = i \int_0^{2\pi} dt \\
&= it \Big|_0^{2\pi} = 2\pi i
\end{aligned}
$$

Consequently,

$$
\boxed{\oint_{C_r} (z - z_0)^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq 0 \end{cases}}
$$

A5) Find the following contour integral.

$$
f(z) = \frac{-2z^2 + 8z - 4}{z^3 - 2z^2 + 2z} = \frac{-2z^2 + 8z - 4}{z(z - 1 + i)(z - 1 - i)}
$$

$$
\begin{aligned}
A &= \lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \frac{-2z^2 + 8z - 4}{z^2 - 2z + 2} \\
&= \frac{-4}{2} = -2
\end{aligned}
$$

$$
\boxed{A = -2}
$$

$$\begin{aligned} B &= \lim_{z\to 1-i}(z-1+i)\,f(z) &&= \lim_{z\to 1-i}\frac{-2z^2+8z-4}{z(z-1-i)} \\ &= \frac{-2(1-i)^2+8(1-i)-4}{(1-i)(1-i-1-i)} &&= \frac{-2(-2i)+8-8i-4}{(1-i)(-2i)} \\ &= \frac{4-4i}{-2-2i} &&= \frac{4-4i}{-2-2i}\cdot\frac{i}{i} \\ &= \frac{4-4i}{2-2i}\cdot i &&= 2i \end{aligned}$$

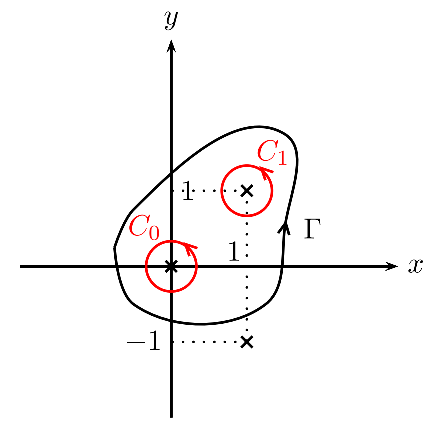
$$\boxed{B = 2i}$$

$$\begin{aligned} C &= \lim_{z\to 1+i}(z-1-i)\,f(z) &&= \lim_{z\to 1+i}\frac{-2z^2+8z-4}{z(z-1+i)} \\ &= \frac{-2(1+i)^2+8(1+i)-4}{(1+i)(1+i-1+i)} &&= \frac{-2(2i)+8+8i-4}{(1+i)(2i)} \\ &= \frac{4+4i}{-2+2i} &&= \frac{4+4i}{-2+2i}\cdot\frac{i}{i} \\ &= \frac{4+4i}{-2-2i}\cdot i &&= -2i \end{aligned}$$

$$\boxed{C = -2i}$$

The singularity; $z=-1+i$, is outside of the positively oriented closed contour; Γ.

$$\begin{aligned} \int_\Gamma \frac{-2z^2+8z-4}{z^3-2z^2+2z}\,dz &= \oint_{C_0}\frac{-2}{z}\,dz + \oint_{C_1}\frac{-2i}{z-1-i} \\ &= 2\pi i\cdot(-2) + 2\pi i\cdot(-2i) \\ &= \boxed{-4\pi i + 4\pi} \end{aligned}$$



A5