TONE MODULATION

$$
\begin{array}{ll}
m(t)=A_{m} \cos \left(2 \pi f_{m} t\right) & : \text { message signal } \\
c(t)=A_{c} \cos \left(2 \pi f_{c} t\right) & : \text { carrier }
\end{array}
$$

Amplitude modulation

$$
\begin{aligned}
s(t) & \left.=\left(1+k_{a} m(t)\right) c(t)\right) \\
& =A_{c}\left(1+\mu \cos \left(2 \pi f_{m} t\right)\right) \cos \left(2 \pi f_{c} t\right)
\end{aligned}
$$

$\mu=k_{a} A_{m}$ is modulation index and $|\mu|<1$. The envelope of $s(t)$ is $A_{c}\left(1+\mu \cos \left(2 \pi f_{m} t\right)\right)$.


Amplitude Modulation. Solid line is the modulated signal and the dotted line is the envelope (message).

Double side band - suppressed carrier

$$
\begin{aligned}
s(t) & =m(t) c(t) \\
& =A_{m} A_{c} \cos \left(2 \pi f_{m} t\right) \cos \left(2 \pi f_{c} t\right) \\
& =\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}-f_{m}\right) t\right)+\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}+f_{m}\right) t\right)
\end{aligned}
$$

Oscillator output at the product demodulator of receiver side is $o(t)$
$o(t)=A_{o} \cos \left(2 \pi f_{c} t+\theta\right)$

$$
\begin{aligned}
y(t) & =s(t) o(t) \\
& =A_{m} A_{c} A_{o} \cos \left(2 \pi f_{m} t\right) \cos \left(2 \pi f_{c} t\right) \cos \left(2 \pi f_{c} t+\theta\right) \\
& =\frac{A_{m} A_{c} A_{o}}{2} \cos (\theta) \cos \left(2 \pi f_{m} t\right)+\frac{A_{m} A_{c} A_{o}}{2} \cos \left(2 \pi f_{m} t\right) \cos \left(4 \pi f_{c} t+\theta\right)
\end{aligned}
$$

The left part of the equation is message signal and the right side is a DSB-SC modulation with carrier frequency of $2 \mathrm{f}_{\mathrm{c}}$. A low pass filter with bandwidth at least equal to the message bandwidth extracts the message signal.

Single side band - suppressed carrier
$s(t)=\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}-f_{m}\right) t\right) \quad$ if lower side band is transmitted
$s(t)=\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}+f_{m}\right) t\right) \quad$ if upper side band is transmitted

$$
\begin{aligned}
y(t) & =s(t) o(t) \\
& =\frac{A_{m} A_{c}}{2} A_{o} \cos \left(2 \pi\left(f_{c} \mp f_{m}\right) t\right) \cos \left(2 \pi f_{c} t+\theta\right) \\
& =\frac{A_{m} A_{c} A_{o}}{2} \cos \left(2 \pi f_{m} t \mp \theta\right)+\frac{A_{m} A_{c} A_{o}}{2} \cos \left(2 \pi\left(2 f_{c} \mp f_{m}\right) t+\theta\right)
\end{aligned}
$$

The left part of the equation is message signal and the right side is the message signal translated to frequency of $2 f_{c}$ in the frequency domain. A low pass filter with bandwidth at least equal to the message bandwidth extracts the message signal.

Single side band with carrier
$s(t)=A_{c} \cos \left(2 \pi f_{c} t\right)+\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}-f_{m}\right) t\right) \quad$ if lower side band is transmitted
$s(t)=A_{c} \cos \left(2 \pi f_{c} t\right)+\frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c}+f_{m}\right) t\right) \quad$ if upper side band is transmitted

$$
\begin{aligned}
s(t) & =A_{c} \cos \left(2 \pi f_{c} t\right)+k_{a} \frac{A_{m} A_{c}}{2} \cos \left(2 \pi\left(f_{c} \mp f_{m}\right) t\right) \\
& =A_{c} \cos \left(2 \pi f_{c} t\right)+\mu A_{c} \cos \left(2 \pi\left(f_{c} \mp f_{m}\right) t\right) \\
& =A_{c} \cos \left(2 \pi f_{c} t\right)+A_{c} \frac{\mu}{2} \cos \left(2 \pi f_{c} t\right) \cos \left(2 \pi f_{m} t\right) \pm A_{c} \frac{\mu}{2} \sin \left(2 \pi f_{c} t\right) \sin \left(2 \pi f_{m} t\right) \\
& =A_{c}\left(1+\frac{\mu}{2} \cos \left(2 \pi f_{m} t\right)\right) \cos \left(2 \pi f_{c} t\right) \pm A_{c} \frac{\mu}{2} \sin \left(2 \pi f_{c} t\right) \sin \left(2 \pi f_{m} t\right)
\end{aligned}
$$

Envelope of $s(t)$ is $s_{e}(t)$ :

$$
\begin{aligned}
s_{e}(t) & =A_{c} \sqrt{\left(1+\frac{\mu}{2} \cos \left(2 \pi f_{m} t\right)\right)^{2}+\frac{\mu^{2}}{4} \sin ^{2}\left(2 \pi f_{c} t\right)} \\
& \approx A_{c}+\frac{\mu}{2} A_{c} \cos \left(2 \pi f_{m} t\right), \quad \text { for } \quad 0<\mu \ll 1
\end{aligned}
$$

## Frequency Modulation

The instantaneous frequency of the carrier:

$$
\begin{aligned}
f_{i}(t) & =f_{c}+k_{f} m(t) \\
& =f_{c}+k_{f} A_{m} \cos \left(2 \pi f_{m} t\right) \\
& =f_{c}+\Delta f \cos \left(2 \pi f_{m} t\right)
\end{aligned}
$$

$\Delta f$ is the frequency deviation. The angle of the carrier:

$$
\begin{aligned}
\theta(t) & =2 \pi \int_{0}^{t} f_{i}(\lambda) d \lambda \\
& =2 \pi f_{c} t+\frac{\Delta f}{f_{m}} \sin \left(2 \pi f_{m} t\right) \\
& =2 \pi f_{c} t+\beta \sin \left(2 \pi f_{m} t\right)
\end{aligned}
$$

$\beta$ is the modulation index.

The modulated signal:

$$
\begin{aligned}
s(t) & =A_{c} \cos (\theta(t)) \\
& =A_{c} \cos \left(2 \pi f_{c} t+\beta \sin \left(2 \pi f_{m} t\right)\right)
\end{aligned}
$$

Narrow band FM $(\beta \ll 1)$

$$
\begin{aligned}
s(t) & =A_{c} \cos \left(2 \pi f_{c} t+\beta \sin \left(2 \pi f_{m} t\right)\right) \\
& =A_{c} \cos \left(2 \pi f_{c} t\right) \cos \left(\beta \sin \left(2 \pi f_{m} t\right)\right)-A_{c} \sin \left(2 \pi f_{c} t\right) \sin \left(\beta \sin \left(2 \pi f_{m} t\right)\right) \\
& \approx A_{c} \cos \left(2 \pi f_{c} t\right)-A_{c} \beta \sin \left(2 \pi f_{m} t\right) \sin \left(2 \pi f_{c} t\right)
\end{aligned}
$$



Frequency Modulation. Solid line is the modulated signal and the dotted line is the message signal.

Wide band FM $(\beta>1)$

Using,

$$
\begin{aligned}
e^{j \beta \sin (x)} & =\sum_{n=-\infty}^{\infty} J_{n}(\beta) e^{j n x} \\
J_{n}(\beta) & =\frac{1}{2 \pi} \int_{-\pi}^{\pi} e^{j(\beta \sin (x)-n x)} d x
\end{aligned}
$$

the FM signal can be rewritten as

$$
\begin{aligned}
s(t) & =A_{c} \cos \left(2 \pi f_{c} t+\beta \sin \left(2 \pi f_{m} t\right)\right) \\
& =A_{c} \sum_{n=-\infty}^{\infty} J_{n}(\beta) \cos \left(2 \pi\left(f_{c}+n f_{m}\right) t\right)
\end{aligned}
$$

Here, $J_{n}(\beta)$ is Bessel function of the first kind of order $n$.

Phase modulation

The angle of the carrier:

$$
\begin{aligned}
\theta(t) & =2 \pi f_{c} t+k_{p} m(t) \\
& =2 \pi f_{c} t+k_{p} A_{m} \cos \left(2 \pi f_{m} t\right) \\
& =2 \pi f_{c} t+\beta \cos \left(2 \pi f_{m} t\right)
\end{aligned}
$$

The instantaneous frequency:

$$
\begin{aligned}
f_{i}(t) & =\frac{1}{2 \pi} \frac{d}{d t} \theta(t) \\
& =f_{c}-\beta f_{m} \sin \left(2 \pi f_{m} t\right) \\
& =f_{c}-\Delta f \sin \left(2 \pi f_{m} t\right)
\end{aligned}
$$

the PM signal
$s(t)=A_{c} \cos \left(2 \pi f_{c} t+\beta \cos \left(2 \pi f_{m} t\right)\right)$

Narrow band PM $(\beta \ll 1)$
$s(t)=A_{c} \cos \left(2 \pi f_{c} t+\beta \cos \left(2 \pi f_{m} t\right)\right)$
$\approx A_{c} \cos \left(2 \pi f_{c} t\right)-A_{c} \beta \cos \left(2 \pi f_{m} t\right) \sin \left(2 \pi f_{c} t\right)$

Wide band PM $(\beta>1)$

Using,

$$
\begin{aligned}
& e^{j \beta \sin (x)}=\sum_{n=-\infty}^{\infty} J_{n}(\beta) e^{j n x} \\
& e^{j \beta \cos (x)}=e^{j \beta \sin \left(x+\frac{\pi}{2}\right)}
\end{aligned}
$$

$$
\begin{aligned}
e^{j \beta \sin \left(x+\frac{\pi}{2}\right)} & =\sum_{n=-\infty}^{\infty} J_{n}(\beta) e^{j n\left(x+\frac{\pi}{2}\right)} \\
& =\sum_{n=-\infty}^{\infty}\left(J_{n}(\beta) e^{j n \frac{\pi}{2}}\right) e^{j n x}
\end{aligned}
$$

the PM signal can be rewritten as

$$
\begin{aligned}
s(t) & =A_{c} \cos \left(2 \pi f_{c} t+\beta \cos \left(2 \pi f_{m} t\right)\right) \\
& =A_{c} \sum_{n=-\infty}^{\infty} J_{n}(\beta) \cos \left(2 \pi\left(f_{c}+n f_{m}\right) t+n \frac{\pi}{2}\right)
\end{aligned}
$$

