Instructions Answer all questions. Give your answers clearly. Each question is worth 25 points. **Time** 80 minutes.
Good Luck.

QUESTIONS

Q1) A signal and its Fourier transform are given in the following.

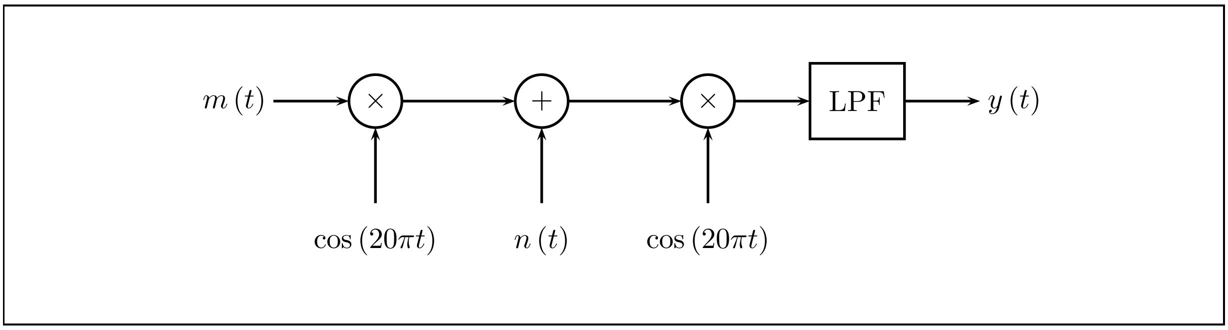
$$m(t) = \frac{1}{1+t^2} \qquad M(f) = \begin{cases} \pi e^{2\pi f}, & f < 0 \\ \pi e^{-2\pi f}, & f > 0 \end{cases}$$

Find Hilbert transform of this signal.

Hint: In the frequency domain Hilbert transform is $-j\,\mathrm{sign}(f)\,M(f)$. Computing inverse Fourier transformation of $-j\,\mathrm{sign}(f)\,M(f)$, Hilbert transform of the signal is obtained.

$$\begin{aligned} \hat{m}(t) &= \int_{-\infty}^{\infty} [-j\,\mathrm{sign}(f)\,M(f)]\, e^{j2\pi f t} df \\ &= j\pi \int_{-\infty}^{0} e^{2\pi f} e^{j2\pi f t} df - j\pi \int_{0}^{\infty} e^{-2\pi f} e^{j2\pi f t} df \\ &= j\pi \int_{-\infty}^{0} e^{(2\pi + j2\pi t) f} df - j\pi \int_{0}^{\infty} e^{(-2\pi + j2\pi t) f} df \\ &= \cdots \end{aligned}$$

Q2) A message signal $m(t) = \cos(3\pi t)$ is modulated via a double side band suppressed carrier modulator. A noise signal $n(t) = \frac{1}{10} \sum_{\ell=-\infty}^{\infty} \delta(t-\ell)$ is added to the modulated signal in the channel. This signal is demodulated by a coherent detector in the receiver (see the block diagram given below). The bandwidth of the low pass filter in the detector is 1.75 Hz. Find $Y(f)$. Compute signal to noise ratio at the output; ratio of the average power of the recovered message signal and the the average power of the noise at the output.



Hint:

$$\sum_{\ell=-\infty}^{\infty} \delta(t-\ell) = \sum_{\ell=-\infty}^{\infty} e^{j2\pi \ell t} = 1 + 2\sum_{\ell=1}^{\infty} \cos(2\pi \ell t)$$

Q3) A message signal, $m(t) = 2\cos(2\pi f_o t) - 6\sin(6\pi f_o t)$ and a carrier, $c(t) = \cos(2\pi f_c t)$ are given. Suppose that $f_c = 10 f_o$. The message signal varies in the range $[-10,\ 10]$.
a) Write AM signal. What should amplitude sensitivity; k_a be chosen for 50% modulation index? Obtain lower and upper side band signals. Plot the spectrum of this AM signal. What is the transmission bandwidth of the AM signal.
b) Write SSB-SC signal for both cases; upper-side band is transmitted and lower side band is transmitted. Obtain the side band signal. Plot the spectrum of this SSB-SC signal. Plot the spectrum of this SSB-SC signal. What is the transmission bandwidth of the SSB-SC signal.

Q4) The following FM signal is given. Find the instantaneous frequency. Find the frequency deviation.

$$s(t) = A\cos\left(2\pi f_o t + 4\frac{f_o}{3}\left(t\,\arctan(t) - \frac{\ln\left(t^2+1\right)}{2}\right) + \frac{\pi}{7}\right)$$

Hint:

$$\frac{d}{dt}\arctan(t) = \frac{1}{t^2+1}$$

$$\frac{d}{dt}\ln(t) = \frac{1}{t}$$

If the range of the instantaneous frequency is $[\,f_c - \Delta f,\ f_c + \Delta f\,]$, Δf is the frequency deviation.

ANSWERS

A1)

$$
\begin{aligned}
\hat{m}(t) &= \int_{-\infty}^{\infty} \left[\, -j \,\mathrm{sign}\,(f)\, M\,(f)\,\right] e^{j2\pi f t} df \\
&= j\pi \int_{-\infty}^{0} e^{2\pi f} e^{j2\pi f t} df - j\pi \int_{0}^{\infty} e^{-2\pi f} e^{j2\pi f t} df \\
&= j\pi \int_{-\infty}^{0} e^{(2\pi + j2\pi t) f} df - j\pi \int_{0}^{\infty} e^{(-2\pi + j2\pi t) f} df \\
&= j\pi \frac{1}{2\pi + j2\pi t} \Big|_{-\infty}^{0} - j\pi \frac{1}{-2\pi + j2\pi t} \Big|_{0}^{\infty} \\
&= j\pi \frac{1}{2\pi + j2\pi t} + j\pi \frac{1}{-2\pi + j2\pi t} \\
&= j\pi \frac{j4\pi t}{-4\pi^2 - 4\pi^2 t^2} \\
&= \frac{-4\pi^2 t}{-4\pi^2 - 4\pi^2 t^2} \\
&= \frac{t}{1 + t^2}
\end{aligned}
$$

A2)

$$
\begin{aligned}
s\,(t) &= m\,(t)\, \cos(20\pi t) &&= \cos(3\pi t)\, \cos(20\pi t) \\
s_1(t) &= s\,(t) + n\,(t) &&= \cos(3\pi t)\, \cos(20\pi t) + \frac{1}{10} + \frac{1}{5} \sum_{\ell=1}^{\infty} \cos(2\pi\, \ell\, t)
\end{aligned}
$$

$$
\begin{aligned}
s_1(t)\, \cos(20\pi t) &= \cos(3\pi t)\, \cos^2(20\pi t) + \frac{1}{10} \cos(20\pi t) + \frac{1}{5} \sum_{\ell=1}^{\infty} \cos(2\pi\, \ell\, t)\, \cos(20\pi t) \\
&= \frac{1}{2} \cos(3\pi t) + \frac{1}{2} \cos(3\pi t) \cos(40\pi t) + \frac{1}{10} \cos(20\pi t) \\
&\quad + \frac{1}{10} \sum_{\ell=1}^{\infty} \cos(20\pi t + 2\pi\, \ell\, t) + \frac{1}{10} \sum_{\ell=1}^{\infty} \cos(20\pi t - 2\pi\, \ell\, t) \\
&= \frac{1}{2} \cos(3\pi t) + \frac{1}{4} \cos(43\pi t) + \frac{1}{4} \cos(37\pi t) + \frac{1}{10} \cos(20\pi t) \\
&\quad + \frac{1}{10} \sum_{\ell=1}^{\infty} \cos(20\pi t + 2\pi\, \ell\, t) + \frac{1}{10} \sum_{\ell=1}^{\infty} \cos(20\pi t - 2\pi\, \ell\, t)
\end{aligned}
$$

$$\begin{aligned} s_1(t)\,\cos(20\pi t) &= \frac{1}{2}\cos(3\pi t) + \frac{1}{4}\cos(43\pi t) + \frac{1}{4}\cos(37\pi t) + \frac{1}{10}\,\cos(20\pi t) \\ &\quad + \frac{1}{10}\sum_{\ell=1}^{\infty}\cos(20\pi t + 2\pi\,\ell\,t) \\ &\quad + \frac{1}{10} + \frac{1}{5}\cos(2\pi t) + \frac{1}{10}\sum_{\substack{\ell=1 \\ \ell\neq 9,\,10,\,11}}^{\infty}\cos(20\pi t - 2\pi\,\ell\,t) \end{aligned}$$

$$y(t) = \frac{1}{10} + \frac{1}{2}\cos(3\pi t) + \frac{1}{5}\cos(2\pi t)$$

$\frac{1}{2}\cos(3\pi t)$ is the message signal recovered in the receiver and $\frac{1}{5}\cos(2\pi t)$ is the unwanted signal (noise). The DC term does not contain any information and can be easily eliminated by a DC block circuit (a series capacitor). The average power of the received message is $\frac{1}{2}\left(\frac{1}{2}\right)^2$ and the average power of the noise is $\frac{1}{2}\left(\frac{1}{5}\right)^2$. Therefore SNR at the output of the receiver is

$$\text{SNR} = \frac{\frac{1}{2}\left(\frac{1}{2}\right)^2}{\frac{1}{2}\left(\frac{1}{5}\right)^2} = \frac{25}{4}$$

A4)

$$\theta(t) = 2\pi f_o t + 4\frac{f_o}{3}\left(t\,\arctan(t) - \frac{\ln\left(t^2+1\right)}{2}\right) + \frac{\pi}{7}$$

$$\begin{aligned} \frac{d}{dt}\theta(t) &= 2\pi f_o + 4\frac{f_o}{3}\left(\arctan(t) + t\,\frac{1}{t^2+1} - \frac{t}{t^2+1}\right) \\ &= 2\pi f_o + 4\frac{f_o}{3}\,\arctan(t) \end{aligned}$$

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi}\frac{d}{dt}\theta(t) \\ &= f_o + \frac{2f_o}{3\pi}\,\arctan(t) \end{aligned}$$

$$\begin{aligned} -\frac{\pi}{2} &\leq \arctan(t) \leq \frac{\pi}{2} \\ f_o - \frac{f_o}{3} &\leq f_i(t) \leq f_o + \frac{f_o}{3} \end{aligned}$$

Hence,

$$\Delta f = \frac{f_o}{3}$$