Power Spectrum.

The power spectrum topic is summarized in the following. Consider that

$$x_T(t) = \begin{cases} x(t), & -T/2 \le t \le T/2\\ 0, & \text{otherwise} \end{cases}$$

The power spectrum of a signal x(t) is defined as

$$S_x(\omega) = \lim_{T \to \infty} \frac{1}{T} |X_T(\omega)|^2 \quad \text{Watt/Hz}$$

where $X_T(\omega) = \mathcal{F}[x_T(t)]$. If x(t) is a random variable then ensemble average of $|X_T(\omega)|^2$ should be used in the definition.

$$S_x(\omega) = \lim_{T \to \infty} \mathsf{E}\left[\frac{1}{T} |X_T(\omega)|^2\right] = \lim_{T \to \infty} \frac{1}{T} \mathsf{E}\left[|X_T(\omega)|^2\right] \quad \text{Watt/Hz}.$$

Recall the Fourier transform property

$$x(t) * y(-t) = \int_{-\infty}^{\infty} x(\lambda) y(\lambda - t) d\lambda$$

$$\mathcal{F}[x(t) * y(-t)] = X(\omega) Y^{*}(\omega).$$

The definition of correlation for power signals is as follows.

$$r_{xy}(t) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(\lambda) y(\lambda - t) d\lambda \text{ for deterministic signals}$$

$$r_{xy}(t) = \lim_{T \to \infty} \frac{1}{T} \mathsf{E} \left[\int_{-T/2}^{T/2} x(\lambda) y(\lambda - t) d\lambda \right] \text{ for random signals}$$

Therefore

$$\mathcal{F}[r_{xy}(t)] = \mathcal{F}\left[\lim_{T \to \infty} \frac{1}{T} \mathsf{E}[x(t) * y(-t)]\right]$$
$$= \lim_{T \to \infty} \frac{1}{T} \mathsf{E}[\mathcal{F}[x(t) * y(-t)]]$$
$$= \lim_{T \to \infty} \frac{1}{T} \mathsf{E}[X(\omega) Y^*(\omega)].$$

For notational simplicity take $r_x(t) = r_{xx}(t)$. Consequently, we get

$$\mathcal{F}[r_x(t)] = \lim_{T \to \infty} \frac{1}{T} \mathsf{E}[X(\omega) X^*(\omega)]$$
$$= S_x(\omega).$$

Suppose that a random signal x(t) is an input to an LTI system with an impulse response h(t). Note that h(t) is deterministic while x(t) is random. The output y(t) is therefore is computed by y(t) = x(t) * h(t). In frequency domain the output is obtained by $Y(\omega) = X(\omega) H(\omega)$. The power spectrum of the output is therefore

$$S_{y}(\omega) = \lim_{T \to \infty} \frac{1}{T} \mathsf{E} \left[|Y(\omega)|^{2} \right] = \lim_{T \to \infty} \frac{1}{T} \mathsf{E} \left[|H(\omega)|^{2} |X(\omega)|^{2} \right]$$
$$= \lim_{T \to \infty} \frac{1}{T} |H(\omega)|^{2} \mathsf{E} \left[|X(\omega)|^{2} \right] = |H(\omega)|^{2} \lim_{T \to \infty} \frac{1}{T} \mathsf{E} \left[|X(\omega)|^{2} \right]$$
$$= |H(\omega)|^{2} S_{x}(\omega)$$

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega)$$

$$x\left(t\right) \longrightarrow h\left(t\right) \longrightarrow y\left(t\right)$$

Noise Analysis in FM Receivers.

In the noise analysis the channel noise w(t) is assumed to white noise. Consider that power spectral density of this noise is $S_x(f) = N_0/2$ Watt/Hz. At the receiver side this noise if filtered with a band-pass filter with a pass band $f_c - B_T/2 < f < f_c + B_T/2$. The filtered noise n(t) is a band-pass signal can be represented by

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t).$$

When message signal is zero the signal at the receiver is

$$\begin{aligned} A_c \cos (2\pi f_c t) + n (t) &= A_c \cos (2\pi f_c t) + n_I (t) \cos (2\pi f_c t) - n_Q (t) \sin (2\pi f_c t) \\ &= (A_c + n_I (t)) \cos (2\pi f_c t) - n_Q (t) \sin (2\pi f_c t) \\ &= r (t) \cos (2\pi f_c t + \psi (t)). \end{aligned}$$

where

$$r(t) = \sqrt{(A_c + n_I(t))^2 + (n_Q(t)\sin(2\pi f_c t))^2}$$

$$\psi(t) = \arctan\left(\frac{n_Q(t)}{A_c + n_I(t)}\right)$$

$$\approx \arctan\left(\frac{n_Q(t)}{A_c}\right) \quad A_c \gg |n(t)| \quad \text{for all } t$$

$$\approx \frac{n_Q(t)}{A_c}$$

Limiter limits the amplitude and makes it constant. The output of the limiter is therefore

$$A\cos\left(2\pi f_{c}t+\psi\left(t\right)\right).$$

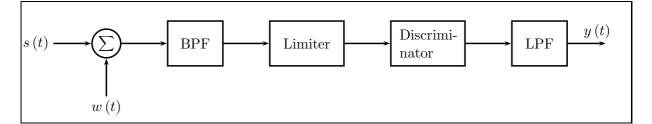
The discriminator output is

$$y(t) = K \frac{d}{dt} \psi(t) = K \frac{1}{A_c} \frac{d}{dt} n_Q(t).$$

When there is no noise in the channel; w(t) = 0, the discriminator output is

$$y(t) = K \frac{d}{dt} \phi(t) = K 2\pi k_f m(t).$$

Note that the analysis is based on the assumption that message signal and the channel noise are independent and that carrier to noise power ratio is quite high.



QUESTIONS

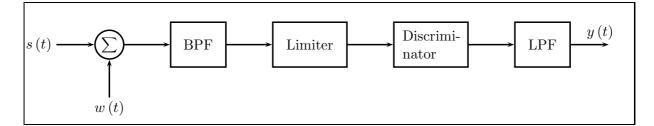
Q1) A message signal $m(t) = \cos(3\pi t)$ is modulated via an frequency modulator. Modulation index of the modulation is $\beta = 2$. A noise

$$w(t) = \frac{1}{20}\cos(18\pi t) + \frac{1}{20}\cos(20\pi t) + \frac{1}{20}\cos(22\pi t)$$

is added to the modulated signal in the channel. This signal is demodulated by a frequency demodulator sketched in the following figure. The bandwidth of the low pass filter in the detector is 2 Hz. Compute signal to noise ratio at the output; ratio of the power of the recovered message signal and the power of the noise at the output.

Assume that the channel noise and the message signal are uncorrelated. Using this assumption noise at the output of the receiver is computed by setting $s(t) = \cos(20\pi t)$, and the recovered message signal is obtained by setting w(t) = 0. Besides these, you may need the following approximation.

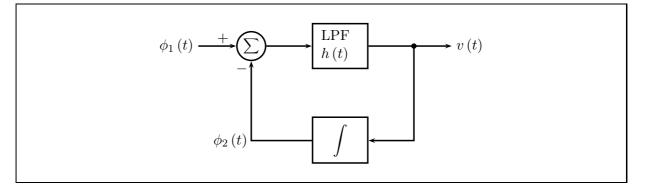
$$A\cos(\theta) + B\cos(\theta + \alpha) \approx C\cos(\theta + \alpha), \qquad A \gg B$$



Q2) The power spectrum of a white noise w(t) is $S_x(\omega) = N_0/2$ quad Watt/Hz. a) Find $r_w(t)$. This white noise is input to an ideal band pass filter with a pass band $f_o - B_T/2 < f < f_o + B_T/2$. The output signal is a band-pass noise. b) Compute the power of n(t). c) Find power spectra of the output noise and its in-phase and quadrature components.

$$w\left(t\right) \xrightarrow{} \begin{array}{c} \text{BPF} \\ h\left(t\right) \end{array} \xrightarrow{} n\left(t\right)$$

Q3) Obtain frequency response of the linear model of PLL shown in the following figure. If $h(t) = \delta(t)$, can the PLL still lock? Write the frequency response of the PLL for $H(f) = \frac{1}{1+j^{\frac{f}{L}}}$.

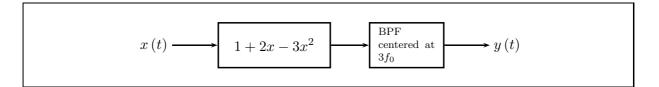


Q4) A superheterodyne receiver is tune to $f_{RF} = 2.09$ MHz frequency. What are the image frequencies that may be received because of imperfect band pass filter at the radio frequency selection section of the receiver?

Q5) Input-output characteristics of a non-linear device is

 $y = 1 + 2x - 3x^2.$

A band pass filter with a pass band $3f_0 - \Delta < f < 3f_0 + \Delta$ is connected in series to this non-linear device, where $0 < \Delta < f_0/2$. For an input $x(t) = \cos(2\pi f_0 t)$, find the signal at the output of the band-pass filter.



Q6)A frequency division multiplexer is given in the following. For the input signals given below plot spectrum of the multiplexer output y(t) (plot Y(f)).

