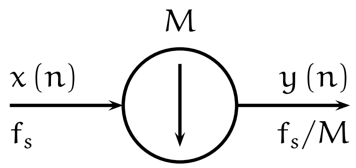
DOWN-SAMPLING



Down-sampling is a process of reducing sampling rate. $M-1$ samples of every M samples are removed.

$$y(n) = x(Mn)$$

While sampling rate of $x(n)$ is f_s, the sampling rate of $y(n)$ is f_s/M. Consider z-transform of $y(n)$.

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(Mn) z^{-n}$$

Therefore

$$Y\left(z^M\right) = \sum_{n=-\infty}^{\infty} x(Mn) z^{-nM}$$

Recall $X(z)$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ &= \cdots + x(Mn) z^{-Mn} + x(Mn+1) z^{-(Mn+1)} + \cdots + x(Mn+M-1) z^{-(Mn+M-1)} + \cdots \\ &= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{M-1} x(Mn+\ell) z^{-(Mn+\ell)} = Y\left(z^M\right) + \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{M-1} x(Mn+\ell) z^{-(Mn+\ell)} \end{aligned}$$

The term at the right-most side should be eliminated to leave $Y\left(z^M\right)$ alone so that we can obtain the z-transform, $Y(z)$. Because

$$\sum_{k=0}^{M-1} W^{\mp k\ell} = \begin{cases} M, & \ell = 0, \mp M, \mp 2M, \mp 3M, \ldots \\ 0, & \text{otherwise} \end{cases}, \qquad \text{where} \quad W = e^{j\frac{2\pi}{M}}$$

and consequently,

$$\begin{aligned}
\sum_{k=0}^{M-1} \left(zW^{-k}\right)^{-Mn-\ell} x(Mn+\ell) &= \sum_{k=0}^{M-1} z^{-Mn-\ell} W^{k\ell} x(Mn+\ell) \\
&= z^{-Mn-\ell} x(Mn+\ell) \sum_{k=0}^{M-1} W^{k\ell} \\
&= \begin{cases} Mz^{-Mn-\ell} x(Mn+\ell), & \ell = 0, \mp M, \mp 2M, \mp 3M, \ldots \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Thus we have the right to write

$$\begin{aligned}
Y\left(z^M\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW^{-k}\right) &= \frac{1}{M}\sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{M-1} x(Mn+\ell)\left(zW^{-k}\right)^{-(Mn+\ell)} \\
&= \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{M-1} x(Mn+\ell)\, z^{-Mn-\ell} \frac{1}{M}\sum_{k=0}^{M-1} W^{k\ell} \\
&= Y\left(z^M\right) + \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{M-1} x(Mn+\ell)\, z^{-Mn-\ell} \frac{1}{M}\sum_{k=0}^{M-1} W^{k\ell} \\
&= Y\left(z^M\right)
\end{aligned}$$

$$\begin{array}{rcll}
X(z) & = & \cdots + & x(Mn+\ell)\, z^{-Mn-\ell} + \cdots \\
X\left(W^{-1}z\right) & = & \cdots + & W^{\ell} x(Mn+\ell)\, z^{-Mn-\ell} + \cdots \\
X\left(W^{-2}z\right) & = & \cdots + & W^{2\ell} x(Mn+\ell)\, z^{-Mn-\ell} + \cdots \\
\cdots & = & \cdots + & \cdots + \cdots \\
X\left(W^{-M+1}z\right) & = & \cdots + & W^{(M-1)\ell} x(Mn+\ell)\, z^{-Mn-\ell} + \cdots \\
+ & & & \\
\hline
M\,Y\left(z^{M}\right) & = & \cdots + & x(Mn+\ell)\, z^{-Mn-\ell} \sum_{k=0}^{M-1} W^{k\ell} + \cdots
\end{array}$$

Finally, we get

$$Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W^{-k}\right)$$

The Fourier transform of $y(n)$ is then

$$Y\left(e^{j\Omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\frac{\Omega-2\pi k}{M}}\right)$$

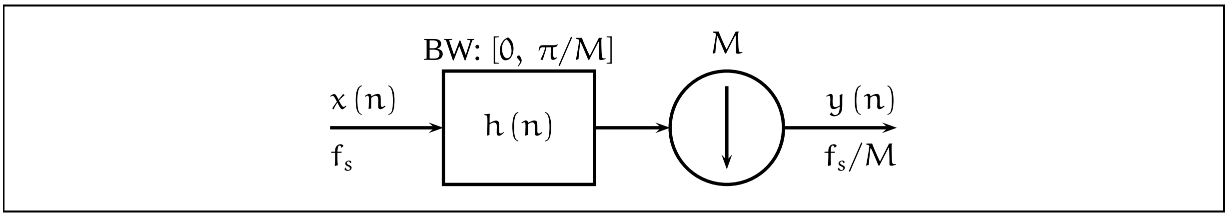
Consider that $X\left(e^{j\Omega}\right)$ is band-limited to α. Then band of $X\left(e^{j\Omega/M}\right)$ will extend to $M\alpha$. To avoid aliasing in $Y\left(e^{j\Omega}\right)$, tail of the image at the right hand side should not exceed $M\alpha$. Since the left limit of this neighboured image (period) is $2\pi - M\alpha$ (the Fourier transform of $Y\left(e^{j\Omega}\right)$ is 2π periodic) we require $M\alpha \leq 2\pi - M\alpha$. Thus $\alpha \leq \pi/M$. The periods of $Y\left(e^{j\Omega}\right)$ will not coincide when $\alpha \leq \pi/M$. Take $L\left(e^{j\Omega}\right) = X\left(e^{j\Omega/M}\right)$. The spectra, $L\left(e^{j(\Omega-2\pi k)}\right) = X\left(e^{j\frac{\Omega-2\pi k}{M}}\right)$, $k = 0, \ldots, M-1$. Consequently,

$$\begin{aligned}
Y\left(e^{j\Omega}\right) &= \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\frac{\Omega-2\pi k}{M}}\right) \\
&= \frac{1}{M}\sum_{k=0}^{M-1} L\left(e^{j(\Omega-2\pi k)}\right)
\end{aligned}$$

As there is no aliasing,

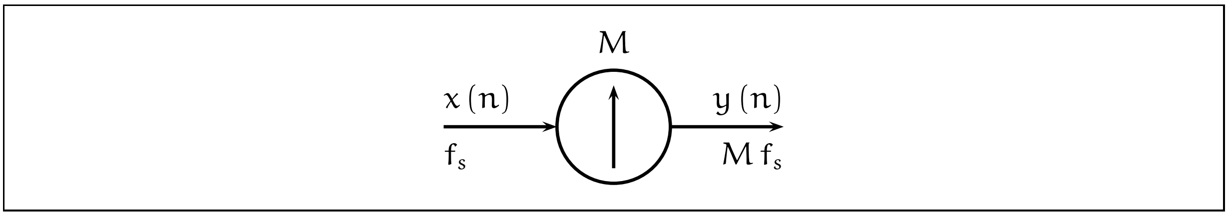
$$Y\left(e^{j\Omega}\right) = \begin{cases} \frac{1}{M} L\left(e^{j\Omega}\right), & -\pi \le \Omega \le \pi \\ Y\left(e^{j(\Omega - 2\pi k)}\right), & k \in \mathbb{Z} \end{cases}$$

In order to avoid aliasing input a signal to be down-sampled has to be band limited by a low pass filter before down-sampling. The low-pass filtering followed by down-sampling is called as decimation.



A Decimator

UP-SAMPLING



Up-sampling is a process of increasing sampling rate. $M-1$ zeros between successive samples are inserted.

$$y(n) = \begin{cases} x(n/M), & n = 0, \pm M, \pm 2M, \pm 3M, \ldots \\ 0, & \text{otherwise} \end{cases}$$

or

$$y(Mn + \ell) = \begin{cases} x(n), & \ell = 0 \\ 0, & \ell \neq 0 \end{cases}$$

While sampling rate of $x(n)$ is f_s, the sampling rate of $y(n)$ is $M f_s$.

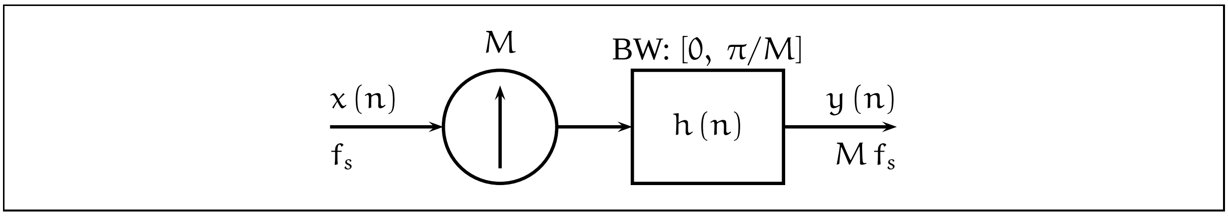
The z-transform of $y(n)$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{M-1} y(Mn+\ell) z^{-Mn-\ell} \\ &= \sum_{n=-\infty}^{\infty} y(Mn) z^{-Mn} + \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^{M-1} y(Mn+\ell) z^{-Mn-\ell} \\ &= \sum_{n=-\infty}^{\infty} x(n) z^{-Mn} = X\left(z^M\right) \end{aligned}$$

The Fourier transform of $y(n)$ follows from the z-transform and it is

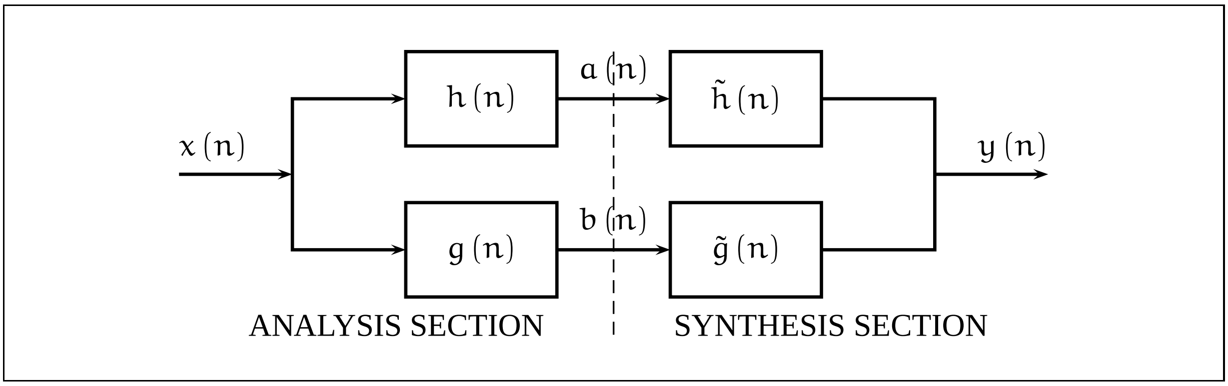
$$Y\left(e^{j\Omega}\right) = X\left(e^{jM\Omega}\right)$$

$X\left(e^{jM\Omega}\right)$ (and hence $Y\left(e^{j\Omega}\right)$) is M-fold compressed version of $X\left(e^{j\Omega}\right)$. Therefore in one period, $Y\left(e^{j\Omega}\right)$ contains M periods of $X\left(e^{j\Omega}\right)$. To leave contents in the frequency band $[-\pi/M,\ \pi/M]$ of the period $[-\pi,\ \pi]$, the images out of this band have to be removed by low-pass filtering. This process is called as interpolation.



An Interpolator

TWO-BAND FILTER BANK WITHOUT UP AND DOWN SAMPLING



Here, $h(n)$ and $\tilde{h}(n)$ are half band low pass filters and $g(n)$ and $\tilde{g}(n)$ are half band high pass filters. We will analyse the filter bank in z-domain. Capital letter of a signal name denotes its z-transform (as $\mathcal{F}[a(n)] = A(z)$). First compute the branch signals

$$A(z) = H(z)X(z)$$
$$B(z) = G(z)X(z)$$

The output is synthesized from the branch signals

$$X(z) = \tilde{H}(z)A(z) + \tilde{G}(z)B(z)$$

Replacing the branch signals with their relations with the input signal we get,

$$\begin{aligned} X(z) &= \tilde{H}(z)(H(z)X(z)) + \tilde{G}(z)(G(z)X(z)) \\ &= \left(H(z)\tilde{H}(z) + G(z)\tilde{G}(z)\right)X(z) \end{aligned}$$

To recover the signal at the output of the synthesis section we require

$$H(z)\tilde{H}(z) + G(z)\tilde{G}(z) = c\,z^{-k}, \qquad c \in \mathbb{R} \quad \text{and} \quad k \in \mathbb{N}\backslash\{0\}$$

because, this yields

$$y(n) = cx(n-k)$$

in the time-domain. We assume that filters are real and finite length. Choosing the synthesis filters as time reverse of the analysis filters may reduce the number of filters needed to realize a filter bank system. Hence,

$$\tilde{H}(z) = z^{-(N-1)} H\left(z^{-1}\right) = \left(z^{N-1} H(z)\right)^{*}$$
$$\tilde{G}(z) = z^{-(N-1)} G\left(z^{-1}\right) = \left(z^{N-1} G(z)\right)^{*}$$

Here, N is the length of the filters. Then,

$$\begin{aligned} H(z)\, z^{-(N-1)} H\left(z^{-1}\right) + G(z)\, z^{-(N-1)} G\left(z^{-1}\right) &= c\, z^{-k} \\ z^{-(N-1)} \left(H(z)\, H\left(z^{-1}\right) + G(z)\, G\left(z^{-1}\right)\right) &= c\, z^{-k} \end{aligned}$$

From the last equation we observe that

$$\begin{aligned} & c = 1, \qquad k = N-1 \\ & H(z)\, H\left(z^{-1}\right) + G(z)\, G\left(z^{-1}\right) = 1 \quad \text{or} \quad H(z)\, H^{*}(z) + G(z)\, G^{*}(z) = 1 \end{aligned}$$

Filters satisfying this relation are called as power complementary filters. A low pass filter can be converted to a high by reflecting its spectrum about the frequency $\pi/2$ (quadrature of 2π). A filter obtained mirroring the spectrum of other filter about $\pi/2$ is called as quadrature-mirror of this filter. Since $g(n)$ is an high-pass filter it can be obtained from the low-pass filter $h(n)$ by computing quadrature-mirror of it. Thus we select,

$$G(z) = -z^{-(N-1)} H\left(-z^{-1}\right)$$

These leads to

$$H(z)\, H\left(z^{-1}\right) + H(-z)\, H\left(-z^{-1}\right) = 1$$

The term

$$R(z) = H(z)\, H\left(z^{-1}\right)$$

is the z-transform of the auto-correlation of $h(n)$

$$r(n) = \sum_{k=-\infty}^{\infty} h(k)\, h(k-n)$$

And henceforth,

$$\begin{aligned} R(z) + R(-z) &= 1 \\ 2\sum_{n=-\infty}^{\infty} r(2n)\, z^{-2n} &= 1 \end{aligned}$$

In the time-domain

$$r(2n) = \sum_{k=-\infty}^{\infty} h(k)\, h(k-2n) = \frac{1}{2}\delta(n)$$

Design Example

Consider that length of the filters is $N = 4$ (filter orders $= N - 1 = 3$) and

$$H(z) = a + bz^{-1} + cz^{-2} + dz^{-3}$$

The others filters are obtained from this filter

$$\begin{aligned} G(z) &= -z^{-3}H\left(-z^{-1}\right) &&= d - cz^{-1} + bz^{-2} - az^{-3} \\ \tilde{H}(z) &= z^{-3}H\left(z^{-1}\right) &&= d + cz^{-1} + bz^{-2} + az^{-3} \\ \tilde{G}(z) &= z^{-3}G\left(z^{-1}\right) &&= -a + bz^{-1} - cz^{-2} + dz^{-3} \end{aligned}$$

It is valuable to impose the constrains; $H(1) = 1$ ($z = 1$ corresponds to $\Omega = 0$), and $H(-1) = 1$ ($z = -1$ corresponds to $\Omega = \pi$), since $h(n)$ is a low-pass filter.

$$\begin{aligned} a + b + c + d &= 1 \\ a - b + c - d &= 0 \end{aligned}$$

From the perfectly reconstruction condition

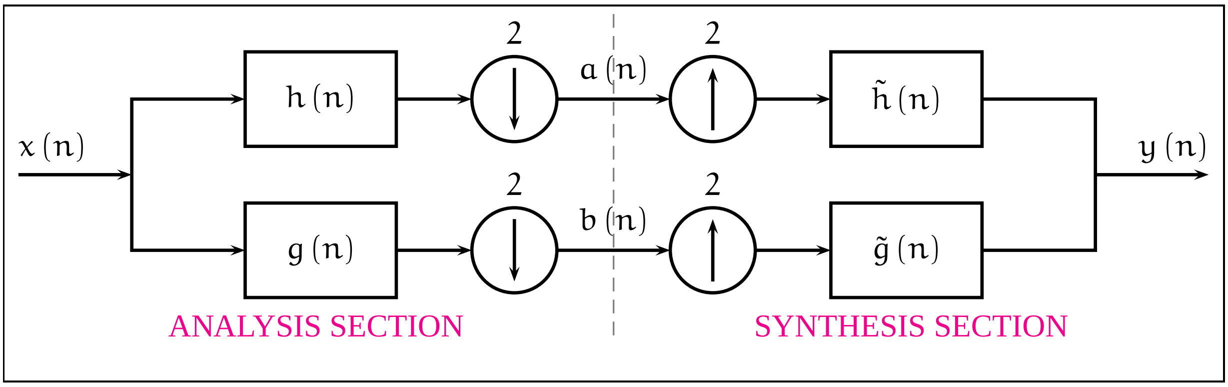
$$\sum_{k=-\infty}^{\infty} h(k)\, h(k-2n) = \frac{1}{2}\delta(n)$$

we get

$$\begin{aligned} a^2+b^2+c^2+d^2 &= \frac{1}{2} \\ a\cdot c+b\cdot d &= 0 \end{aligned}$$

What happens when the analysis filters are replaced with the decimators and the synthesis filters are replaced with the interpolators? What is the condition to obtain perfect reconstruction in this case?

TWO BAND FILTER BANK



In parallel to the above part we will study the filter bank in z-domain. Denote the down-sampling and up-sampling operations as given below.

$$\begin{aligned} [x(n)]_{\downarrow M} &= x(Mn) \\ [X(z)]_{\downarrow M} &= \frac{1}{M}\sum_{k=0}^{M} X\left(z^{1/M}e^{-j2\pi k/M}\right) \end{aligned}$$

down-sampling by a factor M

$$\begin{aligned} [x(n)]_{\uparrow M} &= \sum_{k=-\infty}^{\infty} x(k/M)\,\delta(n-Mk) \\ [X(z)]_{\uparrow M} &= X\left(z^M\right) \end{aligned}$$

up-sampling by a factor M

The branch signals (analysis section outputs) are

$$\begin{aligned} A(z) &= [X(z)H(z)]_{\downarrow 2} = \frac{1}{2}X\left(z^{1/2}\right)H\left(z^{1/2}\right)+\frac{1}{2}X\left(-z^{1/2}\right)H\left(-z^{1/2}\right) \\ B(z) &= [X(z)G(z)]_{\downarrow 2} = \frac{1}{2}X\left(z^{1/2}\right)G\left(z^{1/2}\right)+\frac{1}{2}X\left(-z^{1/2}\right)G\left(-z^{1/2}\right) \end{aligned}$$

And the output of the synthesis section

$$\begin{aligned} Y(z) &= A\left(z^2\right)\tilde{H}(z)+B\left(z^2\right)\tilde{G}(z) \\ &= \left[\frac{1}{2}X(z)H(z)+\frac{1}{2}X(-z)H(-z)\right]\tilde{H}(z) + \left[\frac{1}{2}X(z)G(z)+\frac{1}{2}X(-z)G(-z)\right]\tilde{G}(z) \\ &= \frac{1}{2}X(z)\left[H(z)\tilde{H}(z)+G(z)\tilde{G}(z)\right] + \frac{1}{2}X(-z)\left[H(-z)\tilde{H}(z)+G(-z)\tilde{G}(z)\right] \end{aligned}$$

The most right side of the equation is an outcome of aliasing in the down-sampling process and we require this term to be zero. Therefore

$$\left.\begin{aligned} H(z)\tilde{H}(z)+G(z)\tilde{G}(z) &= 2T(z) \\ H(-z)\tilde{H}(z)+G(-z)\tilde{G}(z) &= 0 \end{aligned}\right. \Rightarrow Y(z)=T(z)X(z)$$

These two equations can be written in matrix form

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix}\begin{bmatrix} \tilde{H}(z) \\ \tilde{G}(z) \end{bmatrix} = 2\begin{bmatrix} T(z) \\ 0 \end{bmatrix}$$

From the last row of the above equation

$$\begin{aligned} H(-z)\tilde{H}(z) &= -G(-z)\tilde{G}(z) \\ \frac{\tilde{H}(z)}{\tilde{G}(z)} &= -\frac{G(-z)}{H(-z)} \end{aligned}$$

Thus, we choose

$$\begin{aligned}\tilde{H}(z) &= G(-z)\\ \tilde{G}(z) &= -H(-z)\end{aligned}$$

And we get

$$H(z)\,G(-z) + (-H(-z))\,G(z) = 2T(z)$$

Let us impose that synthesis filters are time-reverse of the analysis filters. In addition all filters are FIR and causal and length of N.

$$\begin{aligned}\tilde{H}(z) &= G(-z) &= z^{1-N}H\left(z^{-1}\right)\\ \tilde{G}(z) &= -H(-z) &= z^{1-N}G\left(z^{-1}\right)\end{aligned}$$

The above two equations lead to

$$G(z) = -z^{1-N}H\left(-z^{-1}\right), \qquad \text{if } N \text{ is even}$$

We then obtain

$$z^{1-N}\left[H(z)\,H\left(z^{-1}\right) + H(-z)\,H\left(-z^{-1}\right)\right] = 2T(z)$$

Here,

$$R(z) = H(z)\,H\left(z^{-1}\right)$$

is z-transform of the auto-correlation $(\rho(n))$ function of $h(n)$

$$\rho(n) = \sum_{k=-\infty}^{\infty} h(k)\,h(k-n)$$

To reconstruct the signal perfectly we need

$$R(z) + R(-z) = 2 \qquad \left(T(z) = z^{1-N}\right)$$

This requirement yield $y(n) = x(n - N + 1)$. Re-write $R(z)$ and $R(-z)$ in their open form

$$\begin{aligned}
R(z) &= \ldots + \rho(-3)z^3 + \rho(-2)z^2 + \rho(-1)z + \rho(0) + \rho(1)z^{-1} + \rho(2)z^{-2} + \rho(3)z^{-3} + \ldots \\
R(-z) &= \ldots - \rho(-3)z^3 + \rho(-2)z^2 - \rho(-1)z + \rho(0) - \rho(1)z^{-1} + \rho(2)z^{-2} - \rho(3)z^{-3} + \ldots \\
+ & \\
\hline
2 &= \ldots + 0 + 2\rho(-2)z^2 + 0 + 2\rho(0) + 2\rho(2)z^{-2} + 0 + \ldots
\end{aligned}$$

This equality holds if

$$\rho(n) = 0 \quad \text{for } n \text{ is even and } n \neq 0$$

$$\rho(0) = 1$$

Consequently, in time-domain perfectly reconstruction condition

$$\rho(2n) = \sum_{k=-\infty}^{\infty} h(k)\,h(k-2n) = \delta(n)$$

with

$$\begin{aligned}
G(z) &= -z^{1-N}H\left(-z^{-1}\right) &\xrightarrow{\mathcal{Z}^{-1}}\quad & g(n) = -(-1)^n h(N-1-n) \\
\tilde{H}(z) &= z^{1-N}H\left(z^{-1}\right) &\xrightarrow{\mathcal{Z}^{-1}}\quad & \tilde{h}(n) = h(N-1-n) \\
\tilde{G}(z) &= z^{1-N}G\left(z^{-1}\right) &\xrightarrow{\mathcal{Z}^{-1}}\quad & \tilde{g}(n) = g(N-1-n) = (-1)^n h(n)
\end{aligned}$$

Recall the matrix equation

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \begin{bmatrix} \tilde{H}(z) \\ \tilde{G}(z) \end{bmatrix} = 2 \begin{bmatrix} T(z) \\ 0 \end{bmatrix}$$

Replacing

$$\tilde{H}(z) = z^{1-N} H\left(z^{-1}\right)$$

$$\tilde{G}(z) = z^{1-N} G\left(z^{-1}\right)$$

We get

$$z^{1-N} \cdot \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \begin{bmatrix} H\left(z^{-1}\right) \\ G\left(z^{-1}\right) \end{bmatrix} = 2 \begin{bmatrix} T(z) \\ 0 \end{bmatrix}$$

As $T(z) = z^{1-N}$,

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \begin{bmatrix} H\left(z^{-1}\right) \\ G\left(z^{-1}\right) \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This matrix equation depends on z. Now we re-write this matrix equation for $-z$ (replace z by $-z$).

$$\begin{bmatrix} H(-z) & G(-z) \\ H(z) & G(z) \end{bmatrix} \begin{bmatrix} H\left(-z^{-1}\right) \\ G\left(-z^{-1}\right) \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \begin{bmatrix} H\left(-z^{-1}\right) \\ G\left(-z^{-1}\right) \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Combining these two matrix equation we arrive

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \begin{bmatrix} H\left(z^{-1}\right) & H\left(-z^{-1}\right) \\ G\left(z^{-1}\right) & G\left(-z^{-1}\right) \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix

$$\mathbf{F}(z) = \begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix}$$

is called as aliasing cancellation matrix. And

$$\mathbf{F}^{T}\left(z^{-1}\right) = \begin{bmatrix} H\left(z^{-1}\right) & H\left(-z^{-1}\right) \\ G\left(z^{-1}\right) & G\left(-z^{-1}\right) \end{bmatrix}$$

is the inverse matrix. Because $\mathbf{F}^{-1}(z) = \mathbf{F}^{T}\left(z^{-1}\right)$, the matrix $\mathbf{F}(z)$ is called as paraunitary matrix. Using this notation the last matrix equation can be written in more compact form

$$\mathbf{F}(z)\,\mathbf{F}^{T}\left(z^{-1}\right) = 2\mathbf{I}$$

where $\mathbf{I}$ is 2×2 size identity matrix. Adding

$$G(z) = -z^{1-N} H\left(-z^{-1}\right)$$

in the matrix equation we attain

$$\begin{bmatrix} H(z) & -H\left(-z^{-1}\right) \\ H(-z) & H\left(z^{-1}\right) \end{bmatrix} \begin{bmatrix} H\left(z^{-1}\right) & H\left(-z^{-1}\right) \\ -H(-z) & H(z) \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Design Example

Consider that length of the filters is $N = 6$ (filter orders $= N - 1 = 5$) and

$$H(z) = a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5}$$

The others filters are obtained from this filter

$$\begin{aligned} G(z) &= -z^{-5}H\left(-z^{-1}\right) &&= f - ez^{-1} + dz^{-2} - cz^{-3} + bz^{-4} - az^{-5} \\ \tilde{H}(z) &= z^{-5}H\left(z^{-1}\right) &&= f + ez^{-1} + dz^{-2} + cz^{-3} + bz^{-4} + az^{-5} \\ \tilde{G}(z) &= z^{-5}G\left(z^{-1}\right) &&= -a + bz^{-1} - cz^{-2} + dz^{-3} - ez^{-4} + fz^{-5} \end{aligned}$$

From the perfectly reconstruction condition

$$\sum_{k=-\infty}^{\infty} h(k)\, h(k-2n) = \delta(n)$$

we get

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 + e^2 + f^2 &= 1 \\ a \cdot c + b \cdot d + c \cdot e + d \cdot f &= 0 \\ a \cdot e + b \cdot f &= 0 \end{aligned}$$

We have 6 unkowns and 3 equations. We need 3 more equations. It is valuable to impose the constrains; $H(1) = \sqrt{2}$ ($z = 1$ corresponds to $\Omega = 0$), and $H(-1) = 1$ ($z = -1$ corresponds to $\Omega = \pi$), since $h(n)$ is a low-pass filter. In addition we can set the first moment of the filter to be zero to get smoother filter spectrum at $\Omega = 0$.

$$\begin{aligned} a + b + c + d + e + f &= \sqrt{2} \\ a - b + c - d + e - f &= 0 \\ b + 2c + 3d + 4e + 5f &= 0 \end{aligned}$$