QUESTIONS

Q1) Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\Omega n}d\Omega = \begin{cases} 1, & n=0\\ 0, & n\in\mathbb{Z}\setminus\{0\} \end{cases}$$

Q2) Prove that inverse Fourier transform is given by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega n}) e^{j\Omega n} d\Omega$$

Q3) Find z-transform of causal signal $a^{n}u(n)$ and anti-causal signal $b^{n}u(-n-1)$.

Energy signal	:	$\sum_{n} x^{2}(n) < \infty$
Power signal	:	$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}x^{2}\left(n\right)<\infty$
Causal signal	:	$x\left(n\right)=0,\text{for}<0.$
Periodic signal	:	$x\left(n\pm N\right)=x\left(n\right)$

Q4) Let H (z) be a transfer function of a casual LTI system. Compute impulse response h (n) for

a) H (z) =
$$b_0 + b_1 z^{-1}$$

b) H (z) = $\frac{b_0 + b_1 z^{-1}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$ (Find h(n), for n = 0, 1, 2).

Q5) Let H (z) be a transfer function of a casual LTI system and is given as in the following. Compute impulse response h (n) for n = 0 and $n = \infty$ using initial and final value theorems. a) H (z) = $b_0 + b_1 z^{-1}$ b) H (z) = $\frac{b_0 + b_1 z^{-1}}{a_0 + a_1 z^{-1} + a_2 z^{-2}}$. Initial value theorem : $h(0) = \lim_{z \to \infty} H(z)$ Final value theorem : $\lim_{n \to \infty} h(n) = \lim_{z \to 1} (z-1) H(z)$

Q6) Compute the following line integrals for the closed contour |z| = r

$$\oint z^{-1} \mathrm{d} z = ?$$

$$\oint z^{n} dz = ? \qquad n \neq -1$$

Q7) Prove that

$$\frac{1}{2\pi j}\oint X(z) z^{n-1} dz = x(n)$$

Hint:

$$X(z) = \dots + x(k) z^{-k} + \dots + x(n) z^{-n} + \dots$$
$$X(z) z^{n-1} = \dots + x(k) z^{n-k-1} + \dots + x(n) z^{-1} + \dots$$

Q8) Show that for a periodic signal x(n) with period of N

$$\sum_{n=k}^{k+N-1} x(n) = \sum_{n=0}^{N-1} x(n)$$

(The value of the signal in the range { N ... k + N - 1 } for k > 0 is equivalent to the value of the signal in the range { 0 ... k - 1 } or the value of the signal in the range { k ... - 1 } for k < 0 is equivalent to the value of the signal in the range { k + N ... N - 1 } since it is periodic with a period of N).

Q9) Show that for a continuous periodic signal x(t) with period of T

$$\int_{t_{0}-T/2}^{t_{0}+T/2} x(t) dt = \int_{-T/2}^{T/2} x(t) dt$$

(The value of the signal in the range $[T_0/2, t_0 + T_0/2]$ for $t_0 > 0$ is equivalent to the value of the signal in the range $[-T_0/2, t_0 - T_0/2]$ or the value of the signal in the range $[t_0 - T_0/2, -T_0/2]$ for $t_0 < 0$ is equivalent to the value of the signal in the range $[t_0 + T_0/2, T_0/2]$ since it is periodic with a period of T_0).

Q10) Prove the time-shifting property of z-transform.

$$\mathcal{Z}\left[x\left(n-k\right)\right] = z^{-k}X\left(z\right)$$

Q11) Prove the convolution property of z-transform.

$$\mathcal{Z}\left[\mathbf{x}\left(\mathbf{n}\right)*\mathbf{y}\left(\mathbf{n}\right)\right]=\mathbf{X}\left(z\right)\mathbf{Y}\left(z\right)$$

Q12) Show that H (-z) and H (-z⁻¹) are quadrature mirror of H (z). (The filter responses are symmetric about $\Omega = \frac{2\pi}{4} = \frac{\pi}{2}$).

(Consider a bounded (limited in terms of the independent variable) function y(t) in a range [0, T]. g(t) = y(T - t) is a mirror of y(t) at about T/2. In our case the range is $[0, \pi]$ and hence $T = \pi$).

Q13) Two filters H(z) and G(z) are power complementary if they satisfy

$$H(z) H(z^{-1}) + G(z) G(z^{-1}) = 1$$

Consider that G $(z) = H(-z^{-1})$ is quadrature mirror of H (z). Show that this relation is hold

if impulse response of H (z) meets the condition, $\rho(2n) = \frac{1}{2}\delta(n)$ with

$$\rho\left(n\right)=h\left(n\right)*h\left(-n\right)=\sum_{k=-\infty}^{\infty}h\left(k\right)h\left(k-n\right)$$

Q14) Transfer function of a low pass filter is given as

$$H\left(z\right) = b_{0} + b_{1}z^{-1} + b_{2}z^{-2} + b_{3}z^{-3}$$

Consequently, frequency response of the filter

$$H\left(e^{j\Omega}\right)=b_{0}+b_{1}e^{-j\Omega}+b_{2}e^{-2j\Omega}+b_{3}e^{-3j\Omega}$$

Find the filter coefficients for

$$H(1) = 1$$
 $H(j) H(-j) = 1/2$
 $H(-1) = 0$ $H'(1) = 0$

where
$$H'(z) = \frac{d}{dz}H(z)$$
.

Q15) Consider that impulse response of LTI-causal system is $h(n) = a^n u(n)$, with |a| < 1. The output for an arbitrary input x(n) is given by the convolution sum

$$y(n) = x(n) * h(n) = \sum_{k=0}^{\infty} a^{k} x(n-k)$$

Show that this system can be described by the following difference equation.

$$y(n) - ay(n-1) = x(n)$$

Q16) Consider that the following two first order LTI-causal systems are cascaded (connected

in series)

$$S_{I}$$
: $w(n) - aw(n-1) = x(n)$ $x(n)$ is the input, $w(n)$ is the output

 S_{II} : y(n) - by(n-1) = w(n) w(n) is the input, y(n) is the output

Note that $b \neq a$. Show that this cascaded system is equivalent to the following second order LTI-causal system

y(n) - (a + b)y(n - 1) + aby(n - 2) = x(n)

Q17) Consider that the following two first order LTI-causal systems are connected in parallel

$$\begin{array}{lll} S_{I}: & y_{I}\left(n\right) - ay_{I}\left(n-1\right) & = & x\left(n\right) & & x\left(n\right) \text{ is the input, } y_{I}\left(n\right) \text{ is the output} \\ S_{II}: & y_{II}\left(n\right) - by_{II}\left(n-1\right) & = & x\left(n\right) & & x\left(n\right) \text{ is the input, } y_{II}\left(n\right) \text{ is the output} \\ & y\left(n\right) & = & y_{I}\left(n\right) + y_{II}\left(n\right) \end{array}$$

Note that $b \neq a$. Show that this parallel system is equivalent to the following second order LTI-causal system (Hint: $S = S_I + S_{II} - bD^{-1}S_I - aD^{-1}S_{II}$ with D^{-1} is a unit delay operator.).

y(n) - (a + b)y(n - 1) + aby(n - 2) = 2x(n) - (a + b)x(n - 1)

Q18) Find the characteristic equation and the system function of the following second order LTI-causal system. Prove (or show) that the system is stable if the roots of the characteristic equation (equivalently the poles of the system function) are inside the unit circle of the complex plane (Hint: Look Q16. Check impulse responses of the first order systems that build the second order system. When are the stabilities of these first order systems guaranteed ?).

y(n) - (a + b)y(n - 1) + aby(n - 2) = x(n)

Q19) Discrete Fourier transform (DFT) pair of a finite length signal is as in the following

$$X_{k} = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n) \cos(j2\pi kn/N) - j\sum_{n=0}^{N-1} x(n) \sin(j2\pi kn/N)$$
 DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} X_k \cos(j2\pi kn/N) + j \sum_{n=0}^{N-1} X_k \sin(j2\pi kn/N)$$
 IDFT

a) Show that since $X_k = X_{N-k}$, the frequencies $0, 2\pi/N, \dots, \pi - 2\pi/N$ if N is even, or $0, 2\pi/N, \dots, \pi (N-1)/N$ if N is odd, are independent. b) Set $Y_k = \frac{1}{\sqrt{N}} X_k$, and obtain a symmetric transform pair.

$$\begin{split} Y_k &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x\left(n\right) e^{-j2\pi kn/N} & \text{DFT} \\ x\left(n\right) &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Y_k e^{j2\pi kn/N} & \text{IDFT} \end{split}$$

Q20) Discrete cosine transform (DCT) pair of a finite length signal x(n) can be obtained by DFT of the signal y(n) generated from x(n) as in the following

$$y(n) = \begin{cases} x(n), & n = 0, 1, 2, \dots, N-1 \\ x(-n) & n = -N+1, -N+2, \dots, -1 \end{cases}$$

a) Obtain the DCT pair shown in the following.

$$X_{k} = x(0) + 2\sum_{n=1}^{N-1} x(n) \cos(2\pi kn/(2N-1))$$
 DCT

$$x(n) = \frac{1}{2N-1}X_0 + \frac{2}{2N-1}\sum_{k=1}^{N-1}X_k\cos\left(2\pi kn/(2N-1)\right)$$
 IDCT

b) Set $Y_k = \frac{1}{\sqrt{2N-1}}X_k$, and obtain a symmetric transform pair.

$$Y_{k} = \frac{1}{\sqrt{2N-1}} x(0) + \frac{2}{\sqrt{2N-1}} \sum_{n=1}^{N-1} x(n) \cos(2\pi kn/(2N-1))$$
 DCT

$$x\left(n\right) \ = \ \frac{1}{\sqrt{2N-1}}Y_{0} \ + \ \frac{2}{\sqrt{2N-1}}\sum_{k=1}^{N-1}Y_{k}\cos\left(2\pi kn/\left(2N-1\right)\right) \ \ \text{IDCT}$$

Q21) Discrete sine transform (DST) pair of a finite length signal x(n) can be obtained by DFT of the signal y(n) generated from x(n) as in the following

$$y(n) = \begin{cases} x(n-1), & n = 1, 2, ..., N \\ 0, & n = 0 \\ x(-n-1) & n = -N, -N+1, ..., -1 \end{cases}$$

a) Obtain the DST pair shown in the following.

$$X_{k} = 2\sum_{n=0}^{N-1} x(n) \sin(2\pi (k+1) (n+1) / (2N+1))$$
 DST

$$x(n) = \frac{2}{2N+1} \sum_{k=0}^{N-1} X_k \sin \left(2\pi \left(k+1 \right) \left(n+1 \right) / \left(2N+1 \right) \right)$$
 IDST

b) Set $Y_k = \frac{1}{\sqrt{2N+1}}X_k$, and obtain a symmetric transform pair.

$$Y_{k} = \frac{2}{\sqrt{2N+1}} \sum_{n=0}^{N-1} x(n) \sin(2\pi (k+1) (n+1) / (2N+1))$$
 DST

$$x(n) = \frac{2}{\sqrt{2N+1}} \sum_{k=0}^{N-1} Y_k \sin(2\pi (k+1) (n+1) / (2N+1))$$
 IDST