Instructions Answer all questions. Give your answers clearly. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Laplace transform is bi-lateral if it is not stated otherwise. Calculator and cell phone are not allowed in the exam. Each question is worth 15 points. Time 135 minutes.


## QUESTIONS

Q1) For the signal given below plot a) $-x(t / 3)$,
b) $x(t+2), \quad$ c) $x(t-1)$


Q1
Q2) Compute the convolution integrals; a) $e^{-t} u(t) * e^{-2 t} u(t), \quad$ b) $e^{-t} u(t) * e^{-t} u(t)$.
Q3) Find the impulse response of the following causal-LTI system. Do all computations in time domain.

$$
\frac{d^{2}}{d t^{2}} y(t)+5 \frac{d}{d t} y(t)+6 y(t)=\frac{d}{d t} x(t)+4 x(t)
$$

Q4) Find the impulse response of the following causal-LTI system. Do all computations in time domain.

$$
y[n]-\frac{5}{6} y[n-1]+\frac{1}{6} y[n-2]=x[n]-\frac{2}{3} x[n-1]
$$

Q5) Compute trigonometric Fourier series coefficients of the following periodic signal. Note that the signal is even; $x(t)=x(-t)$ and its mean is zero. Observe that the coefficients of the cosine terms can be computed by employing symmetry of the signal;

$$
C_{k}=\frac{2}{T} \int_{-T / 2}^{T / 2} x(t) \cos \left(\frac{2 \pi}{T} k t\right) d t=\frac{4}{T} \int_{0}^{T / 2} x(t) \cos \left(\frac{2 \pi}{T} k t\right) d t
$$

Q6)
a) Compute Fourier transform of $e^{(-1+2 j) t} u(t)$ by using the transformation integral.
b) Compute inverse Fourier transform of $X(\omega)$ by employing partial fraction expansion.

$$
X(\omega)=\frac{2 j \omega+2}{(j \omega)^{2}+2 j \omega+5}=\frac{2 j \omega+2}{(j \omega+1-2 j)(j \omega+1+2 j)}
$$

Q7)
a) Find frequency response of the following causal-LTI system by using Fourier transformation (do all computations in the frequency domain).

$$
\frac{d^{2}}{d t^{2}} y(t)+4 \frac{d}{d t} y(t)+4 y(t)=-\frac{d}{d t} x(t)-3 x(t)
$$


b) Find the impulse response by computing inverse Fourier transform of the frequency response. Employ indirect method; partial fraction expansion to compute the inverse Fourier transform.

Q8)
a) Compute Laplace transform of $e^{(-1+2 j) t} u(t)$ by using the transformation integral.
b) Compute inverse Laplace transform of $X(s)$ by employing partial fraction expansion.

$$
X(s)=\frac{2 s+2}{s^{2}+2 s+5}=\frac{2 s+2}{(s+1-2 j)(s+1+2 j)}, \quad \operatorname{Re}[s]>-1
$$

Q9) Transfer function of a causal-LTI system has two poles; $s=-1$ and $s=-3$ and one zero; $s=-2$. The value of the system response at $s=0$ is $H(0)=2$.
a) Determine and plot the region of convergence. b) Obtain the transfer function. c) Compute the impulse response.

Q10) a) Find system response of the following causal-LTI system by using Laplace transformation.

$$
\frac{d^{2}}{d t^{2}} y(t)+3 \frac{d}{d t} y(t)+2 y(t)=2 \frac{d}{d t} x(t)+9 x(t)
$$

b) Find the impulse response by computing inverse Laplace transform of the system response. Employ indirect method; partial fraction expansion to compute the inverse Laplace transform.
c) The initial values of the impulse response are given as $h\left(0^{+}\right)=2$ and $h^{\prime}\left(0^{+}\right)=3$. Obtain the system response by employing uni-lateral Laplace transformation.

## Table of Laplace Transforms

| Signal | Laplace transform |  |
| :--- | :--- | :--- |
| $e^{-a t} u(t)$ | $\frac{1}{s+a}$, | $\operatorname{Re}[s]>-\operatorname{Re}[a]$ |
| $-e^{-a t} u(-t)$ | $\frac{1}{s+a}$, | $\operatorname{Re}[s]<-\operatorname{Re}[a]$ |
| $t e^{-a t} u(t)$ | $\frac{1}{(s+a)^{2}}$, | $\operatorname{Re}[s]>-\operatorname{Re}[a]$ |

ANSWERS

A1)

$$
y(t)=-x(t / 3) \Rightarrow y(3 t)=-x(t)
$$

$-x(t)$ moves to $3 t$ in the graph of $y(t)$.

a)

$$
y(t)=x(t+2) \Rightarrow y(t-2)=x(t)
$$

$x(t)$ moves to $t-2$ in the graph of $y(t)$.

b)

$$
y(t)=x(t-1) \Rightarrow y(t+1)=x(t)
$$

$x(t)$ moves to $t+1$ in the graph of $y(t)$.
A2)

$$
\begin{aligned}
e^{-t} u(t) * e^{-2 t} u(t) & =\int_{-\infty}^{\infty} e^{-\lambda} u(\lambda) e^{-2(t-\lambda)} u(t-\lambda) d \lambda \\
& = \begin{cases}\int_{0}^{t} e^{-\lambda} e^{-2(t-\lambda)} d \lambda, & t>0 \\
0, & t<0\end{cases}
\end{aligned}
$$


c)

$$
\int_{0}^{t} e^{-\lambda} e^{-2(t-\lambda)} d \lambda=e^{-2 t} \int_{0}^{t} e^{\lambda} d \lambda=e^{-t}-e^{-2 t}
$$

Hence,

$$
e^{-t} u(t) * e^{-2 t} u(t)=\left\{\begin{array}{ll}
e^{-t}-e^{-2 t}, & t>0 \\
0, & t<0
\end{array}=e^{-t} u(t)-e^{-2 t} u(t)\right.
$$

Similarly,

$$
\begin{aligned}
e^{-t} u(t) * e^{-t} u(t)=\int_{-\infty}^{\infty} e^{-\lambda} u(\lambda) e^{-(t-\lambda)} u(t-\lambda) d \lambda & = \begin{cases}e^{-t} \int_{0}^{t} d \lambda, & t>0 \\
0, & t<0\end{cases} \\
& =t e^{-t} u(t)
\end{aligned}
$$

A3) In the solution of this question we will need the first and the second derivatives of function $g(t)=$ $A e^{a t} u(t)$;

$$
\begin{aligned}
g^{\prime}(t) & =A a e^{a t} u(t)+A \delta(t) \\
g^{\prime \prime}(t) & =A a^{2} e^{a t} u(t)+A a \delta(t)+A \delta^{\prime}(t)
\end{aligned}
$$

When input is an impulse; $x(t)=\delta(t)$ the output is the impulse response; $y(t)=h(t)$. Therefore we should find the solution of the following differential equation.

$$
\frac{d^{2}}{d t^{2}} h(t)+5 \frac{d}{d t} h(t)+6 h(t)=\frac{d}{d t} \delta(t)+4 \delta(t)
$$

Because system is causal the impulse response, $h(t)$ is zero for $t<0$. As the impulse disappears after $t=0$ we get a homogenous differential equation

$$
\frac{d^{2}}{d t^{2}} h(t)+5 \frac{d}{d t} h(t)+6 h(t)=0, \quad t>0
$$

There is a non-zero response for $t>0$ and can be calculated from the homogenous differential equation. The characteristic equation of the differential equation is

$$
D^{2}+5 D+6=(D+2)(D+3)=0
$$

The roots of this equation are $D_{1}=-2$ and $D_{2}=-3$. Since the roots are distinct the impulse response contains two exponents for $t>0$. For all $t$, it is as in the following.

$$
h(t)=\left\{\begin{array}{ll}
0, & t<0 \\
A e^{-2 t}+B e^{-3 t}, & t>0
\end{array}=A e^{-2 t} u(t)+B e^{-3 t} u(t)\right.
$$

here $A$ and $B$ are unknown coefficients can be easily determined from the differential equation . Now, we compute the first and the second derivatives of the impulse response and replace them in the differential equation to find the unknown coefficients.

$$
\begin{aligned}
h^{\prime}(t) & =-2 A e^{-2 t} u(t)-3 B e^{-3 t} u(t)+(A+B) \delta(t) \\
h^{\prime \prime}(t) & =4 A e^{-2 t} u(t)+9 B e^{-3 t} u(t)-(2 A+3 B) \delta(t)+(A+B) \delta^{\prime}(t)
\end{aligned}
$$

When the impulse response and its derivatives are replaced in the differential equation exponential terms will vanish because the exponents in the impulse response is the solution of the homogenous differential equation. Then,

$$
\begin{array}{lll}
-(2 A+3 B) \delta(t)+(A+B) \delta^{\prime}(t)+5(A+B) \delta(t)+6 \cdot 0 & = & \delta^{\prime}(t)+4 \delta(t) \\
(3 A+2 B) \delta(t)+(A+B) \delta^{\prime}(t) & = & \delta^{\prime}(t)+4 \delta(t)
\end{array}
$$

The coefficients of $\delta^{\prime}(t)$ and $\delta(t)$ in the both side of the equation should be equal if the equality holds.

$$
\begin{aligned}
& 3 A+2 B=4 \\
& A+B=1
\end{aligned}
$$

The unknown coefficients are easily extracted from the above equations.

$$
\begin{array}{ll}
3 A+2 B & =4 \\
A+2(A+B) & =4 \\
A+2 \cdot 1 & =4 \\
A & =2 \\
& =1 \\
A+B & =1 \\
2+B & =-1
\end{array}
$$

Consequently, we get the impulse response.

$$
h(t)=2 e^{-2 t} u(t)-e^{-3 t} u(t)
$$

The coefficients $A$ and $B$ could also be computed from the initials conditions extracted from the differential equation. We first compute double integral of the differential equation over the interval $\left[0^{-}, 0^{+}\right]$ to find $h\left(0^{+}\right)$and later we compute integral of the differential equation over the interval $\left[0^{-}, 0^{+}\right]$to find $h^{\prime}\left(0^{+}\right)$. This approach is valid when the impulse response does not have an impulse at the origin. Since the highes derivative of the input is less than the highest derivative of the output in the differential equation the impulse response does not have an impulse at the origin. Therefore,

$$
\begin{array}{ll}
\left.h(t)\right|_{0^{-}} ^{0^{+}}+\left.5 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}}+\left.6 \mathcal{D}^{-2} h(t)\right|_{0^{-}} ^{0^{+}} & =\left.u(t)\right|_{0^{-}} ^{0^{+}}+\left.4 t u(t)\right|_{0^{-}} ^{0^{+}} \\
h\left(0^{+}\right)-h\left(0^{-}\right)+5 \cdot 0+6 \cdot 0 & \\
h\left(0^{+}\right) & =u\left(0^{+}\right)-u\left(0^{-}\right)+4 \cdot 0 \\
& =1
\end{array}
$$

$$
\begin{array}{ll}
\left.h^{\prime}(t)\right|_{0^{-}} ^{0^{+}}+\left.5 h(t)\right|_{0^{-}} ^{0^{+}}+\left.6 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}} & =\left.\delta(t)\right|_{0^{-}} ^{0^{+}}+\left.4 u(t)\right|_{0^{-}} ^{0^{+}} \\
h^{\prime}\left(0^{+}\right)-h^{\prime}\left(0^{-}\right)+5\left(h\left(0^{+}\right)-h\left(0^{-}\right)\right)+6 \cdot 0 & =\delta\left(0^{+}\right)-\delta\left(0^{-}\right)+4\left(u\left(0^{+}\right)-u\left(0^{-}\right)\right) \\
h^{\prime}\left(0^{+}\right)+5 & =4 \\
h^{\prime}\left(0^{+}\right) & =-1 \\
& \\
h\left(0^{+}\right)=A+B \quad=1 \\
h^{\prime}\left(0^{+}\right)=-2 A-3 B=-1 \\
& \\
2 A+3 B=1 \\
2+B & =1 \\
B & =-1 \\
A+B & =1 \\
A+-1 & =1 \\
A & =2
\end{array}
$$

This approach can be generalized. Consider the following second order differential equation.

$$
\frac{d^{2}}{d t^{2}} y(t)+a_{1} \frac{d}{d t} y(t)+a_{2} y(t)=b_{0} \frac{d}{d t} x(t)+b_{1} x(t)
$$

The initial conditions of the impulse response for this differential equation are then

$$
\begin{aligned}
& h\left(0^{+}\right)=b_{0} \\
& h^{\prime}\left(0^{+}\right)=b_{1}-a_{1} b_{0}
\end{aligned}
$$

A4) In the solution of this question we will need the first and the second difference of function $g[n]=$ $A a^{n} u[n]$;

$$
\left.\begin{array}{rl}
g[n-1]=A a^{n-1} u[n-1] & =A a^{-1} a^{n}(u[n]-\delta[n]) \\
& =A a^{-1} a^{n} u[n]-A a^{-1} \delta[n] \\
g[n-2] & =A a^{-1} a^{n-1} u[n-1]-A a^{-1} \delta[n-1]
\end{array}\right)=A a^{-2} a^{n} u[n]-A a^{-2} \delta[n]-A a^{-1} \delta[n-1] .
$$

When input is an impulse; $x[n]=\delta[n]$ the output is the impulse response; $y[n]=h[n]$. Therefore we should find the solution of the following differential equation.

$$
h[n]-\frac{5}{6} h[n-1]+\frac{1}{6} h[n-2]=\delta[n]-\frac{2}{3} \delta[n-1]
$$

Because system is causal the impulse response, $h[n]$ is zero for $n<0$. As the impulses at the right side of the difference equation disappear after $n=1$ we get a homogenous differential equation

$$
h[n]-\frac{5}{6} h[n-1]+\frac{1}{6} h[n-2]=0, \quad n>1
$$

There is a non-zero response for $n \geq 0$ and can be calculated from the homogenous differential equation. The characteristic equation of the differential equation is

$$
1-\frac{5}{6} D^{-1}+\frac{1}{6} D^{-2}=\left(1-\frac{1}{2} D^{-1}\right)\left(1-\frac{1}{3} D^{-1}\right)=0
$$

The roots of this equation are $D_{1}=\frac{1}{2}$ and $D_{2}=\frac{1}{3}$. Since the roots are distinct the impulse response contains two exponents for $n \geq 0$. For all $n$, it is as in the following.

$$
h[n]=\left\{\begin{array}{ll}
0, & n<0 \\
A\left(\frac{1}{2}\right)^{n}+B\left(\frac{1}{3}\right)^{n}, & n \geq 0
\end{array}=A\left(\frac{1}{2}\right)^{n} u[n]+B\left(\frac{1}{3}\right)^{n} u[n]\right.
$$

here $A$ and $B$ are unknown coefficients can be easily determined from the first difference equations. Now, we compute the first and the second differences of the impulse response and replace them in the difference equation to find the unknown coefficients.

$$
\begin{aligned}
& h[n-1]=2 A\left(\frac{1}{2}\right)^{n} u[n]+3 B\left(\frac{1}{3}\right)^{n} u[n]-(2 A+3 B) \delta[n] \\
& h[n-2]=4 A\left(\frac{1}{2}\right)^{n} u[n]+9 B\left(\frac{1}{3}\right)^{n} u[n]-(2 A+3 B) \delta[n-1]-(4 A+9 B) \delta[n]
\end{aligned}
$$

When the impulse response and its difference are replaced in the difference equation exponential terms will vanish because the exponents in the impulse response is the solution of the homogenous difference equation. Then,

$$
\begin{array}{ll}
0-\frac{5}{6}(-2 A-3 B) \delta[n]+\frac{1}{6}(-(2 A+3 B) \delta[n-1]-(4 A+9 B) \delta[n]) & =\delta[n]-\frac{2}{3} \delta[n-1] \\
(A+B) \delta[n]-\frac{1}{6}(2 A+3 B) \delta[n-1] & =\delta[n]-\frac{2}{3} \delta[n-1]
\end{array}
$$

The coefficients of $\delta[n-1]$ and $\delta[n]$ in the both side of the equation should be equal if the equality holds.

$$
\begin{array}{ll}
A+B & =1 \\
2 A+3 B & =4
\end{array}
$$

The unknown coefficients are easily extracted from the above equations.

$$
\begin{array}{ll}
2 A+3 B & =4 \\
2(A+B)+B & =4 \\
2 \cdot 1+B & =4 \\
B & =2 \\
& =1 \\
A+B & =1 \\
A+2 & =-1
\end{array}
$$

Consequently, we get the impulse response.

$$
h[n]=2\left(\frac{1}{2}\right)^{n} u[n]-\left(\frac{1}{3}\right)^{n} u[n]
$$

The coefficients $A$ and $B$ could also be computed from the initials conditions extracted from the difference equation. We first compute the difference equation for $n=0$ and later for $n=1$. Therefore,
$h[0]-\frac{5}{6} h[-1]+\frac{1}{6} h[-2]=\delta[0]-\frac{2}{3} \delta[-1]$
$h[0]-\frac{5}{6} \cdot 0+\frac{1}{6} \cdot 0=1-\frac{2}{3} \cdot 0$
$h[0]=1$

$$
3 A+2 B=1
$$

$$
A+2=1
$$

$$
A \quad=-1
$$

$$
A+B=1
$$

$$
-1+B=1
$$

$$
B \quad=2
$$

This approach can be generalized. Consider the following second order difference equation.

$$
y[n]+a_{1} y[n-1]+a_{2} y[n-2]=b_{0} x[n]+b_{1} x[n-1]
$$

The initial conditions of the impulse response for this difference equation are then

$$
\begin{aligned}
h[0] & =b_{0} \\
h[1] & =b_{1}-a_{1} b_{0}
\end{aligned}
$$

A5) The fundamental period of the periodic signal is $T=3$ seconds. Hence,


$$
\begin{aligned}
& h[1]-\frac{5}{6} h[0]+\frac{1}{6} h[-1] \quad=\delta[1]-\frac{2}{3} \delta[0] \\
& h[1]-\frac{5}{6} \cdot 1+\frac{1}{6} \cdot 0=0-\frac{2}{3} \cdot 1 \\
& h[1]-\frac{5}{6} \quad=-\frac{2}{3} \\
& h[1] \quad=\frac{5}{6}-\frac{2}{3} \\
& h[1] \quad=\frac{1}{6} \\
& \begin{aligned}
h[0] & =A+B=1 \\
h[1] & =\frac{1}{2} A+\frac{1}{3} B=\frac{1}{6} \\
& =3 A+2 B=1
\end{aligned}
\end{aligned}
$$

$$
\begin{aligned}
C_{k} & =\frac{4}{T} \int_{0}^{T / 2} x(t) \cos \left(\frac{2 \pi}{T} k t\right) d t \\
& =\frac{4}{3} \int_{0}^{0.5} 2 \cos \left(\frac{2 \pi}{3} k t\right) d t-\frac{4}{3} \int_{0.5}^{1.5} \cos \left(\frac{2 \pi}{3} k t\right) d t \\
& =\left.\frac{4}{3} \cdot 2 \cdot \frac{3}{2 \pi k} \sin \left(\frac{2 \pi}{3} k t\right)\right|_{0} ^{0.5}-\left.\frac{4}{3} \cdot \frac{3}{2 \pi k} \sin \left(\frac{2 \pi}{3} k t\right)\right|_{0.5} ^{1.5} \\
& =\frac{4}{\pi k}\left[\sin \left(\frac{\pi}{3} k\right)-0\right]-\frac{2}{\pi k}\left[\sin (\pi k)-\sin \left(\frac{\pi}{3} k\right)\right] \\
& =\frac{4}{\pi k} \sin \left(\frac{\pi}{3} k\right)-\frac{2}{\pi k}\left[0-\sin \left(\frac{\pi}{3} k\right)\right] \\
& =\frac{2}{\pi k} \sin \left(\frac{\pi}{3} k\right), \\
& =0=1,2, \cdots, \infty \\
C_{3 k} & =\frac{2}{\pi(3 k-1)}(-1)^{k+1} \frac{\sqrt{3}}{2} \\
C_{3 k-1} & =\frac{2}{\pi(3 k-2)}(-1)^{k+1} \frac{\sqrt{3}}{2}, \quad k=1,2, \cdots, \infty
\end{aligned}
$$

A6)
a)

$$
\begin{aligned}
\mathcal{F}\left[e^{(-1+2 j) t} u(t)\right] & =\int_{-\infty}^{\infty} e^{(-1+2 j) t} u(t) e^{-j \omega t} d t \\
& =\int_{0}^{\infty} e^{(-1+2 j-j \omega) t} d t \\
& =\left.\frac{1}{-1+2 j-j \omega} e^{(-1+2 j-j \omega) t}\right|_{0} ^{\infty}
\end{aligned}
$$

Consider, $e^{(-1+2 j-j \omega) t}$. It is a complex number and its magnitude and phase are computed in the following.

$$
\begin{aligned}
e^{(-1+2 j-j \omega) t} & =e^{-t} e^{j(2-\omega) t} \\
& =A e^{j \theta}
\end{aligned}
$$

The following limits are needed to compute the transform.

$$
\begin{array}{lll}
A=e^{-t} & \text { and } & \theta=(2-\omega) t \\
\lim _{t \rightarrow \infty} A \rightarrow 0 & \text { therefore } & \lim _{t \rightarrow \infty} A e^{j \theta} \rightarrow 0
\end{array}
$$

Next,

$$
\begin{aligned}
\left.\frac{1}{-1+2 j-j \omega} e^{(-1+2 j-j \omega) t}\right|_{0} ^{\infty} & =\frac{1}{-1+2 j-j \omega}\left[\lim _{t \rightarrow \infty} e^{(-1+2 j-j \omega) t}-1\right] \\
& =\frac{1}{-1+2 j-j \omega}[0-1] \\
& =\frac{1}{1-2 j+j \omega}
\end{aligned}
$$

Consequently,

$$
\mathcal{F}\left[e^{(-1+2 j) t} u(t)\right]=\frac{1}{1-2 j+j \omega}
$$

b)

$$
\begin{aligned}
2 j \omega+2 & =j \omega+1-2 j+j \omega+1+2 j \\
\frac{2 j \omega+2}{(j \omega+1-2 j)(j \omega+1+2 j)} & =\frac{j \omega+1-2 j+j \omega+1+2 j}{(j \omega+1-2 j)(j \omega+1+2 j)} \\
& =\frac{j \omega+1-2 j}{(j \omega+1-2 j)(j \omega+1+2 j)}+\frac{j \omega+1+2 j}{(j \omega+1-2 j)(j \omega+1+2 j)} \\
& =\frac{1}{j \omega+1+2 j}+\frac{1}{j \omega+1-2 j}
\end{aligned}
$$

$$
\mathcal{F}^{-1}\left[\frac{1}{1+2 j+j \omega}\right]=e^{-(1+2 j) t} u(t)
$$

$$
\mathcal{F}^{-1}\left[\frac{1}{1-2 j+j \omega}\right]=e^{-(1-2 j) t} u(t)
$$

$$
\mathcal{F}^{-1}[X(\omega)] \quad=\mathcal{F}^{-1}\left[\frac{1}{1+2 j+j \omega}\right]+\mathcal{F}^{-1}\left[\frac{1}{1-2 j+j \omega}\right]
$$

$$
=e^{-(1+2 j) t} u(t)+e^{-(1-2 j) t} u(t)
$$

$$
=e^{-t} \underbrace{\left(e^{-j 2 t}+e^{j 2 t}\right)}_{2 \cos (2 t)} u(t)
$$

$$
=2 e^{-t} \cos (2 t) u(t)
$$

A7)
a) Fourier transformation of the differential equation yields

$$
\begin{aligned}
(j \omega)^{2} Y(\omega)+4 j \omega Y(\omega)+4 Y(\omega) & =-j \omega X(\omega)-3 X(\omega) \\
H(\omega) & =\frac{Y(\omega)}{X(\omega)} \\
& =-\frac{j \omega+3}{(j \omega)^{2}+4 j \omega+4} \\
& =-\frac{j \omega+3}{(j \omega+2)^{2}}
\end{aligned}
$$

b)

$$
\begin{aligned}
& \begin{aligned}
& \frac{j \omega+3}{(j \omega+2)^{2}}= \frac{j \omega+2+1}{(j \omega+2)^{2}} \\
&= \frac{j \omega+2}{(j \omega+2)^{2}}+\frac{1}{(j \omega+2)^{2}} \\
&= \frac{1}{j \omega+2}+\frac{1}{(j \omega+2)^{2}} \\
&=-\frac{1}{j \omega+2}-\frac{1}{(j \omega+2)^{2}} \\
& H(\omega)
\end{aligned} \\
& \begin{aligned}
& \mathcal{F}^{-1}[H(\omega)]-\mathcal{F}^{-1}\left[\frac{1}{j \omega+2}\right]-\mathcal{F}^{-1}\left[\frac{1}{(j \omega+2)^{2}}\right] \\
&=e^{-2 t} u(t) \\
& \mathcal{F}^{-1}\left[\frac{1}{j \omega+2}\right]
\end{aligned} \\
& \mathcal{F}^{-1}\left[\frac{1}{(j \omega+2)^{2}}\right] \\
& =t^{-2 t} u(t) \\
& \mathcal{F}^{-1}[H(\omega)]=-\mathcal{F}^{-1}\left[\frac{1}{j \omega+2}\right]-\mathcal{F}^{-1}\left[\frac{1}{(j \omega+2)^{2}}\right]
\end{aligned}
$$

A8)
a)

$$
\begin{aligned}
\mathcal{F}\left[e^{(-1+2 j) t} u(t)\right] & =\int_{-\infty}^{\infty} e^{(-1+2 j) t} u(t) e^{-s t} d t \\
& =\int_{0}^{\infty} e^{(-1+2 j-s) t} d t \\
& =\left.\frac{1}{-1+2 j-s} e^{(-1+2 j-s) t}\right|_{0} ^{\infty}
\end{aligned}
$$

Consider, $e^{(-1+2 j-s) t}$. It is a complex number and its magnitude and phase are computed in the following. Replace complex variable $s$ with its cartesian form; $s=\sigma+j \omega$.

$$
\begin{aligned}
e^{(-1+2 j-s) t}=e^{(-1+2 j-\sigma-j \omega) t} & =e^{(-1-\sigma) t} e^{j(2-\omega) t} \\
& =A e^{j \theta}
\end{aligned}
$$

The following limits are needed to compute the transform.

$$
\begin{array}{lll}
A=e^{(-1-\sigma) t} & \text { and } & \theta=(2-\omega) t \\
\lim _{t \rightarrow \infty} A= \begin{cases}0, & \sigma>-1 \\
\infty & \sigma<-1\end{cases} & \text { therefore } & \lim _{t \rightarrow \infty} A e^{j \theta}= \begin{cases}0, & \sigma>-1 \\
\infty & \sigma<-1\end{cases}
\end{array}
$$

Next,

$$
\begin{aligned}
\left.\frac{1}{-1+2 j-s} e^{(-1+2 j-s) t}\right|_{0} ^{\infty} & =\frac{1}{-1+2 j-s}\left[\lim _{t \rightarrow \infty} e^{(-1+2 j-s) t}-1\right] \\
& =\frac{1}{-1+2 j-s}[0-1], \quad \text { when } \sigma>-1(\sigma=\operatorname{Re}[s]) \\
& =\frac{1}{1-2 j+s}
\end{aligned}
$$

Consequently,

$$
\mathcal{L}\left[e^{(-1+2 j) t} u(t)\right]=\frac{1}{1-2 j+s}, \quad \operatorname{Re}[s]>-1
$$

b)

$$
\begin{aligned}
2 s+2 & =s+1-2 j+s+1+2 j \\
\frac{2 s+2}{(s+1-2 j)(s+1+2 j)} & =\frac{s+1-2 j+s+1+2 j}{(s+1-2 j)(j \omega+1+2 j)} \\
& =\frac{s+1-2 j}{(s+1-2 j)(s+1+2 j)}+\frac{s+1+2 j}{(s+1-2 j)(s+1+2 j)} \\
& =\frac{1}{s+1+2 j}+\frac{1}{s+1-2 j}
\end{aligned}
$$

$$
\begin{aligned}
e^{-(1+2 j) t} u(t) & \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{1+2 j+s}, \quad \operatorname{Re}[s]>-1 \\
e^{-(1-2 j) t} u(t) & \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{1-2 j+s}, \quad \operatorname{Re}[s]>-1 \\
& =\mathcal{L}^{-1}\left[\frac{1}{1+2 j+s}\right]+\mathcal{L}^{-1}\left[\frac{1}{1-2 j+s}\right] \\
& =e^{-(1+2 j) t} u(t)+e^{-(1-2 j) t} u(t) \\
& =e^{-t} \underbrace{\left(e^{-j 2 t}+e^{j 2 t}\right)}_{2 \cos (2 t)} u(t) \\
& =2 e^{-t} \cos (2 t) u(t)
\end{aligned}
$$

A9)
a) Since the system is causal, the region of convergence is the right side of a vertical line containing the right most pole in the complex plane. The region of convergence is then $\operatorname{Re}[s]>-1$.

b)

$$
\begin{aligned}
& H(s)=K \frac{s+2}{(s+1)(s+3)} \\
& H(0)=K \frac{2}{3}=3 \\
&=3 \\
& H(s)=3 \frac{s+2}{(s+1)(s+3)}
\end{aligned}
$$

c) First, we extend the transfer function into its partial fractions.

$$
H(s)=\frac{A}{s+1}+\frac{B}{s+3}
$$

$$
\begin{aligned}
A & =\lim _{s \rightarrow-1}(s+1) H(s) \\
A & =\lim _{s \rightarrow-1}(s+1) 3 \frac{s+2}{(s+1)(s+3)} \\
A & =\lim _{s \rightarrow-1} 3 \frac{s+2}{(s+3)} \\
A & =\frac{3}{2}
\end{aligned}
$$

$$
B=\lim _{s \rightarrow-3}(s+3) H(s)
$$

$$
B=\lim _{s \rightarrow-3}(s+3) 3 \frac{s+2}{(s+1)(s+3)}
$$

$$
B=\lim _{s \rightarrow-3} 3 \frac{s+2}{(s+1)}
$$

$$
B=\frac{3}{2}
$$

$$
H(s)=\frac{3}{2} \frac{1}{s+1}+\frac{3}{2} \frac{1}{s+3}
$$

$$
\begin{aligned}
h(t) & =\mathcal{L}^{-1}[H(s)] \\
& =\frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]+\frac{3}{2} \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] \\
e^{-t} u(t) & \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \operatorname{Re}[s]>-1 \\
e^{-3 t} u(t) & \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+3}, \quad \operatorname{Re}[s]>-3 \\
h(t) & =\frac{3}{2} e^{-t} u(t)+\frac{3}{2} e^{-3 t} u(t)
\end{aligned}
$$

A10)
a) Laplace transformation of the differential equation yields

$$
s^{2} Y(s)+3 s Y(s)+2 Y(s)=2 s X(s)+9 X(s)
$$

$$
\begin{aligned}
H(s)=\frac{Y(s)}{X(s)} & =\frac{2 s+9}{s^{2}+3 s+2} \\
& =\frac{2 s+9}{(s+1)(s+2)}, \quad \operatorname{Re}[s]>-1
\end{aligned}
$$

b) We extend the transfer function into its partial fractions.

$$
H(s)=\frac{A}{s+1}+\frac{B}{s+2}
$$

$$
\begin{aligned}
A & =\lim _{s \rightarrow-1}(s+1) H(s) \\
A & =\lim _{s \rightarrow-1}(s+1) \frac{2 s+9}{(s+1)(s+2)} \\
A & =\lim _{s \rightarrow-1} \frac{2 s+9}{(s+2)} \\
A & =7
\end{aligned}
$$

$$
B=\lim _{s \rightarrow-2}(s+2) H(s)
$$

$$
B=\lim _{s \rightarrow-2}(s+2) \frac{2 s+9}{(s+1)(s+2)}
$$

$$
B=\lim _{s \rightarrow-2} \frac{2 s+9}{(s+1)}
$$

$$
B=-5
$$

$$
H(s)=\frac{7}{s+1}-\frac{5}{s+2}
$$

$$
\begin{aligned}
& h(t)= \\
& \mathcal{L}^{-1}[H(s)] \\
&=7 \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]-5 \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\
& e^{-t} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \operatorname{Re}[s]>-1 \\
& e^{-2 t} u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \operatorname{Re}[s]>-2 \\
& h(t) \\
&=7 e^{-t} u(t)-5 e^{-2 t} u(t)
\end{aligned}
$$

c) The system response can also be obtained by employing unilateral Laplace transformation. We first re-write the differential equation for the impulse exerted to the input; when $x(t)=\delta(t), y(t)=h(t)$ and then we compute unilateral Laplace transformation of the resultant differential equation.

$$
H(s)=\frac{A}{s+1}+\frac{B}{s+2}
$$

$$
\begin{array}{ll}
h^{\prime \prime}(t)+3 h^{\prime}(t)+2 h(t) & =2 \delta^{\prime}(t)+9 \delta(t) \\
\mathcal{U} \mathcal{L}\left[h^{\prime \prime}(t)+3 h^{\prime}(t)+2 h(t)\right] & =\mathcal{U} \mathcal{L}\left[2 \delta^{\prime}(t)+9 \delta(t)\right] \\
\mathcal{U} \mathcal{L}\left[h^{\prime \prime}(t)\right]+3 \mathcal{U} \mathcal{L}\left[h^{\prime}(t)\right]+2 \mathcal{U} \mathcal{L}[h(t)] & =2 \mathcal{U} \mathcal{L}\left[\delta^{\prime}(t)\right]+9 \mathcal{U L}[\delta(t)]
\end{array}
$$

$$
\mathcal{U L}\left[h^{\prime}(t)\right]=s H(s)-h\left(0^{+}\right) \quad=s H(s)-2
$$

$$
\mathcal{U} \mathcal{L}\left[h^{\prime \prime}(t)\right]=s^{2} H(s)-\operatorname{sh}\left(0^{+}\right)-h^{\prime}\left(0^{+}\right)=s^{2} H(s)-2 s-3
$$

$$
\mathcal{U} \mathcal{L}[\delta(t)]=0
$$

$$
\mathcal{U} \mathcal{L}\left[\delta^{\prime}(t)\right]=0
$$

Furthermore,

$$
\begin{array}{ll}
\mathcal{U} \mathcal{L}\left[h^{\prime \prime}(t)\right]+3 \mathcal{U} \mathcal{L}\left[h^{\prime}(t)\right]+2 \mathcal{U L}[h(t)] & =2 \mathcal{U} \mathcal{L}\left[\delta^{\prime}(t)\right]+9 \mathcal{U} \mathcal{L}[\delta(t)] \\
s^{2} H(s)-2 s-3+3(s H(s)-2)+2 H(s) & =0 \\
\left(s^{2}+3 s+2\right) H(s)-2 s-9 & =0 \\
\left(s^{2}+3 s+2\right) H(s) & =2 s+9 \\
H(s) & =\frac{2 s+9}{s^{2}+3 s+2}
\end{array}
$$

