Instructions Answer all questions. Give your answers clearly. Laplace transform is bi-lateral if the otherwise is not stated. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Each question is worth 25 points. **Time** 135 minutes.

1	2	3	4	5	TOTAL

QUESTIONS

Q1) An electrical circuit is given as in the following. Designate the capacitor voltage, $v_C(t)$ and the inducer current, $i_L(t)$ as the state variables. Obtain state-form for the input-output relation.

Q2) For the following signals, compute a) $\int_{-\infty}^{\infty} a^2(t)\,dt$, b) $\int_{-\infty}^{\infty} b^2(t)\,dt$ and, c) $\int_{-\infty}^{\infty} a(t)\,b(t)\,dt$.

d) Find and plot

$$x(t) = \frac{1}{\sqrt{3}}a\left(\frac{t}{2}\right) - \frac{1}{2\sqrt{3}}a\left(\frac{t}{2}+1\right) + \frac{1}{\sqrt{3}}b\left(\frac{t}{2}-1\right)$$

Q3) A linear-time invariant system is formed as in the following block diagram. Find the output, $y(t)$ for the input, $x(t)$ given below.

Q4) The positive linear feedback system is sketched in the following figure. $A(s)$ and $B(s)$ are the system functions of their corresponding system blocks and $X(s)$ and $Y(s)$ are the input and the output respectively, in the s-domain.

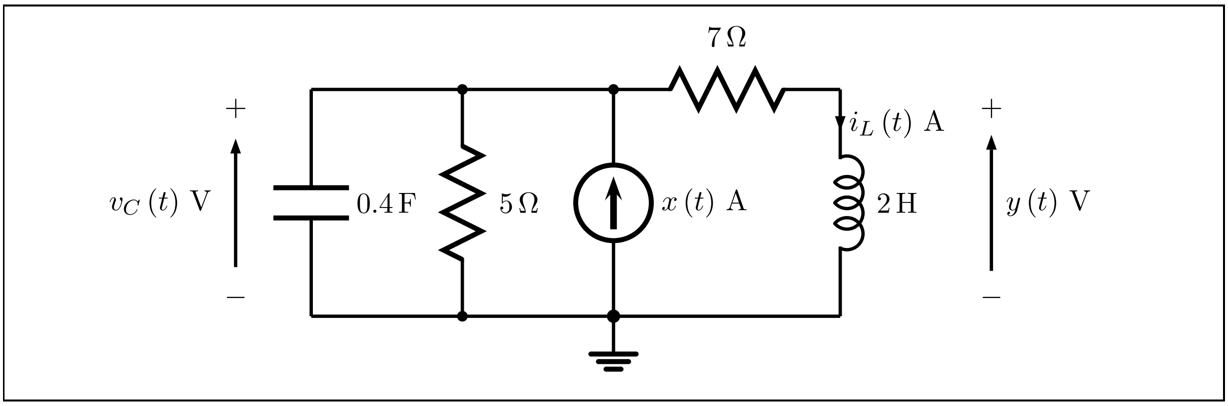
a) Obtain the system function, $H(s)$, of the feedback system in terms of $A(s)$ and $B(s)$.

b) Re-write $H(s)$ for $A(s) = 8$ and $B(s) = \frac{1}{s^2+9}$. Find roots of $1 - A(s)B(s) = 0$. Find $h(t) = \mathsf{L}^{-1}[H(s)]$.

c) Re-write $H(s)$ for $A(s) = -9$ and $B(s) = \frac{1}{s+3}$. Find $h(t) = \mathsf{L}^{-1}[H(s)]$.

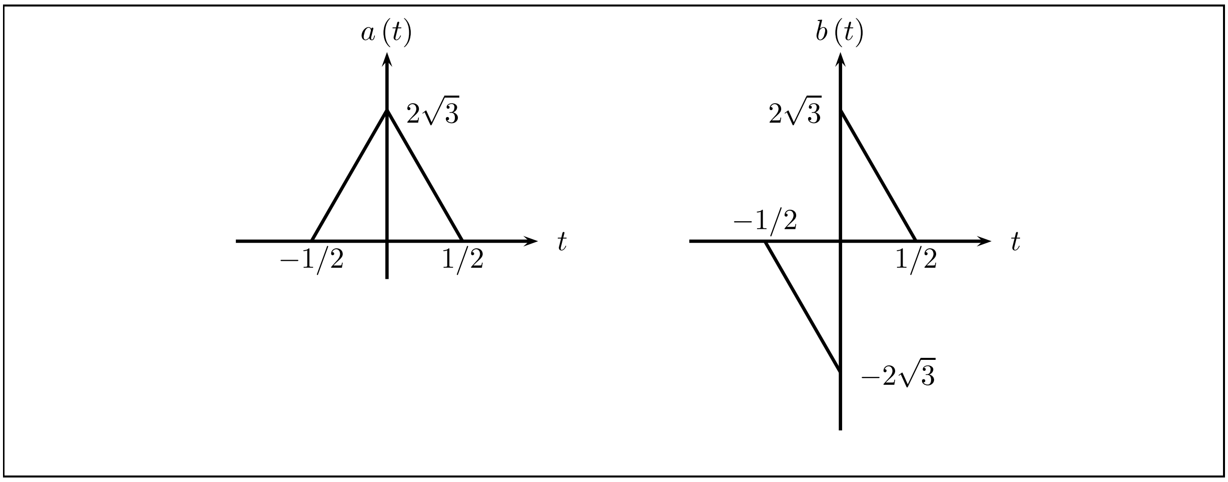
d) Repeat c) for $A(s) = -2$ and $B(s) = \frac{1}{s+2}$.

Q5) An electrical circuit is given as in the following. a) Obtain the differential equation describing the input-output relation of the circuit (system). b) Find the frequency response of the system. c) Plot the magnitude and phase spectra (compute the values for the critical points $\omega = -\infty, -1, 0, 1, \infty$ in order to plot the spectra). d) Compute the impulse response of the system (expand the frequency response into partial fractions to obtain the inverse of the frequency response).



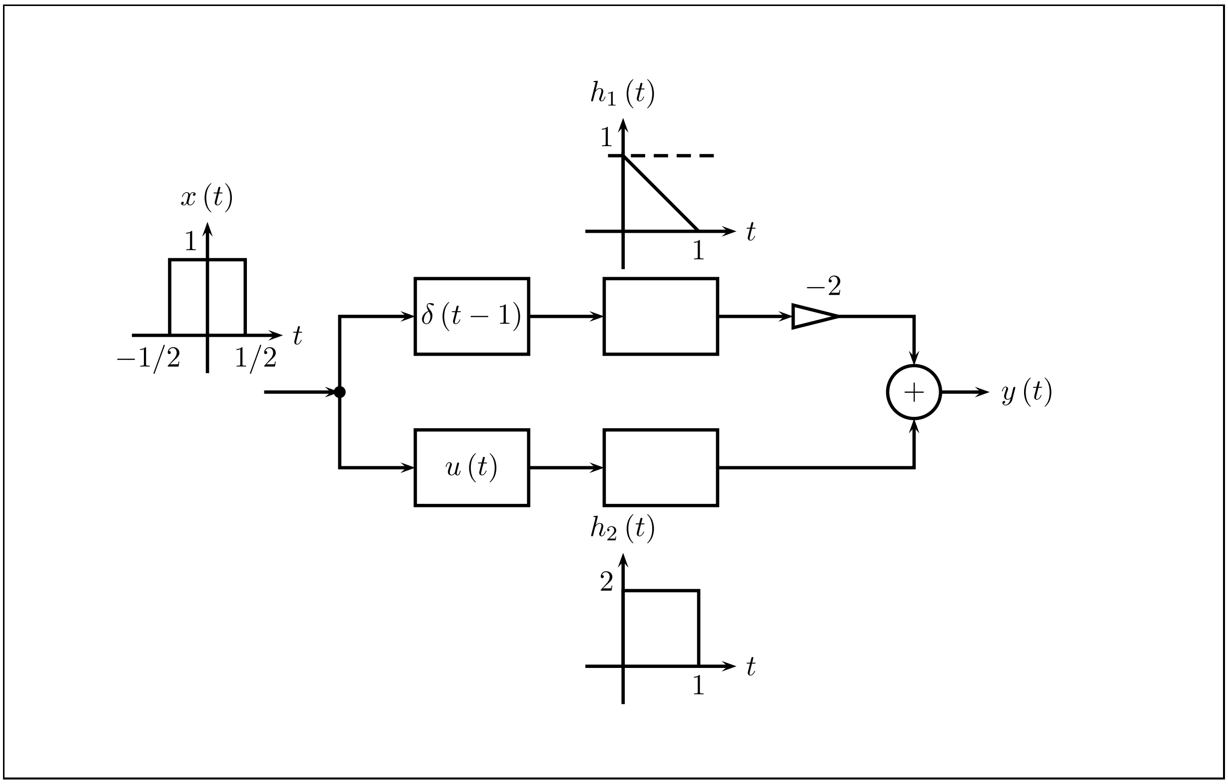
7 Ω
$i_L(t)$ A
+
+
$v_C(t)$ V
0.4 F
5 Ω
$x(t)$ A
2 H
$y(t)$ V
−
−

Q1



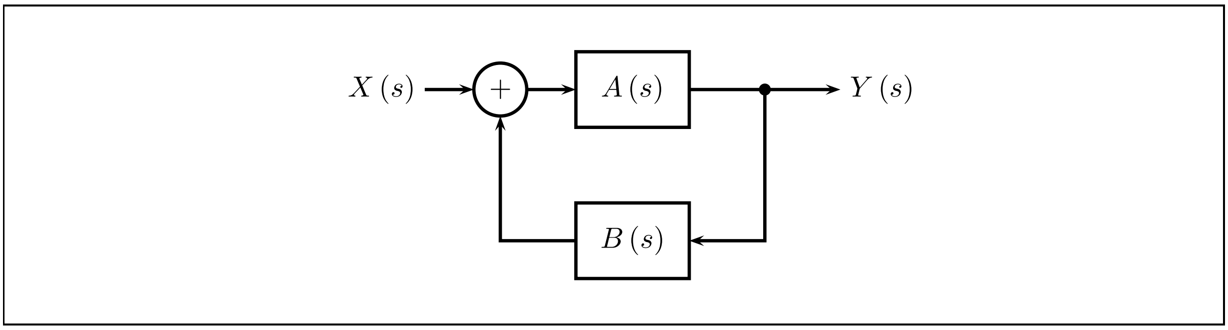
$a(t)$
$2\sqrt{3}$
t
$-1/2$
$1/2$
$b(t)$
$2\sqrt{3}$
$-1/2$
t
$1/2$
$-2\sqrt{3}$

Q2



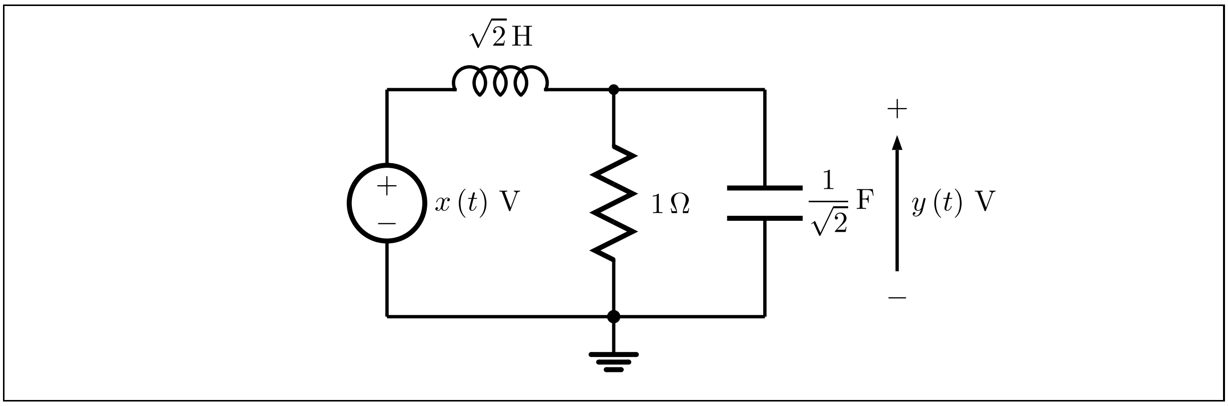
$h_1(t)$
1
$x(t)$
1
t
1
$-1/2$
$1/2$
t
-2
$\delta(t-1)$
$y(t)$
$u(t)$
$h_2(t)$
2
t
1

Q3



$X(s)$
$A(s)$
$Y(s)$
$B(s)$

Q4



$\sqrt{2}$ H
$x(t)$ V
$1\,\Omega$
$\frac{1}{\sqrt{2}}$ F
$y(t)$ V

Q5

Table of Laplace Transforms

Signal	Laplace transform	
$\delta(t)$	1	
$u(t)$	$\frac{1}{s}$,	$\mathrm{Re}[s] > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$,	$\mathrm{Re}[s] > -\mathrm{Re}[a]$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$,	$\mathrm{Re}[s] < -\mathrm{Re}[a]$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$,	$\mathrm{Re}[s] > -\mathrm{Re}[a]$
$\cos(at)\,u(t)$	$\frac{s}{s^2+a^2}$,	$\mathrm{Re}[s] > 0$
$\sin(at)\,u(t)$	$\frac{a}{s^2+a^2}$,	$\mathrm{Re}[s] > 0$
$e^{-at}\cos(bt)\,u(t)$	$\frac{s+a}{(s+a)^2+b^2}$,	$\mathrm{Re}[s] > -\mathrm{Re}[a]$
$e^{-at}\sin(bt)\,u(t)$	$\frac{b}{(s+a)^2+b^2}$,	$\mathrm{Re}[s] > -\mathrm{Re}[a]$

ANSWERS

A1) By employing Kirchhoff's current law at the node we get

$$\begin{aligned} 0.4v_C'(t) + \frac{v_C(t)}{5} - x(t) + i_L(t) &= 0 \\ -0.5v_C(t) + 2.5x(t) - 2.5i_L(t) &= v_C'(t) \end{aligned}$$

And by writing Kirchhoff's voltage law for the outer loop we get

$$\begin{aligned} 7i_L(t) + 2i_L'(t) - v_C(t) &= 0 \\ -3.5i_L(t) + 0.5v_C(t) &= i_L'(t) \end{aligned}$$

From the outer loop, the output signal $y(t)$ can be easily obtained.

$$y(t) = v_C(t) - 7i_L(t)$$

The state-form is obtained by re-writing the equations collectively, in matrix form.

$$\begin{aligned} \begin{bmatrix} v_C'(t) \\ i_L'(t) \end{bmatrix} &= \begin{bmatrix} -0.5, & -2.5 \\ 0.5, & -3.5 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 2.5 \\ 0 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1, & -7 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} \end{aligned}$$

Table of Fourier Transforms

Signal	Fourier transform
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$e^{-at}u(t)$	$\frac{1}{j\omega + a}$
$-e^{-at}u(-t)$	$-\frac{1}{j\omega + a}$
$te^{-at}u(t)$	$\frac{1}{(j\omega + a)^2}$
$\cos(at)\,u(t)$	$\frac{j\omega}{(j\omega)^2 + a^2} + \frac{\pi}{2}\left[\delta(\omega - a) + \delta(\omega + a)\right]$
$\sin(at)\,u(t)$	$\frac{a}{(j\omega)^2 + a^2} + \frac{\pi}{2j}\left[\delta(\omega - a) - \delta(\omega + a)\right]$
$e^{-at}\cos(bt)\,u(t)$	$\frac{j\omega + a}{(j\omega + a)^2 + b^2}$
$e^{-at}\sin(bt)\,u(t)$	$\frac{b}{(j\omega + a)^2 + b^2}$

A2)

$$a(t) = \begin{cases} 4\sqrt{3}\,(t + 1/2)\,, & -1/2 < t < 0 \\ 4\sqrt{3}\,(t - 1/2)\,, & 0 < t < 1/2 \\ 0, & \text{otherwise} \end{cases} \qquad b(t) = \begin{cases} -4\sqrt{3}\,(t + 1/2)\,, & -1/2 < t < 0 \\ 4\sqrt{3}\,(t - 1/2)\,, & 0 < t < 1/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} a(-t) &= a(t) & \qquad b(-t) &= -b(t) \\ a^2(t) &= b^2(t) & \qquad a(-t)\,b(-t) &= -a(t)\,b(t) \end{aligned}$$

a)

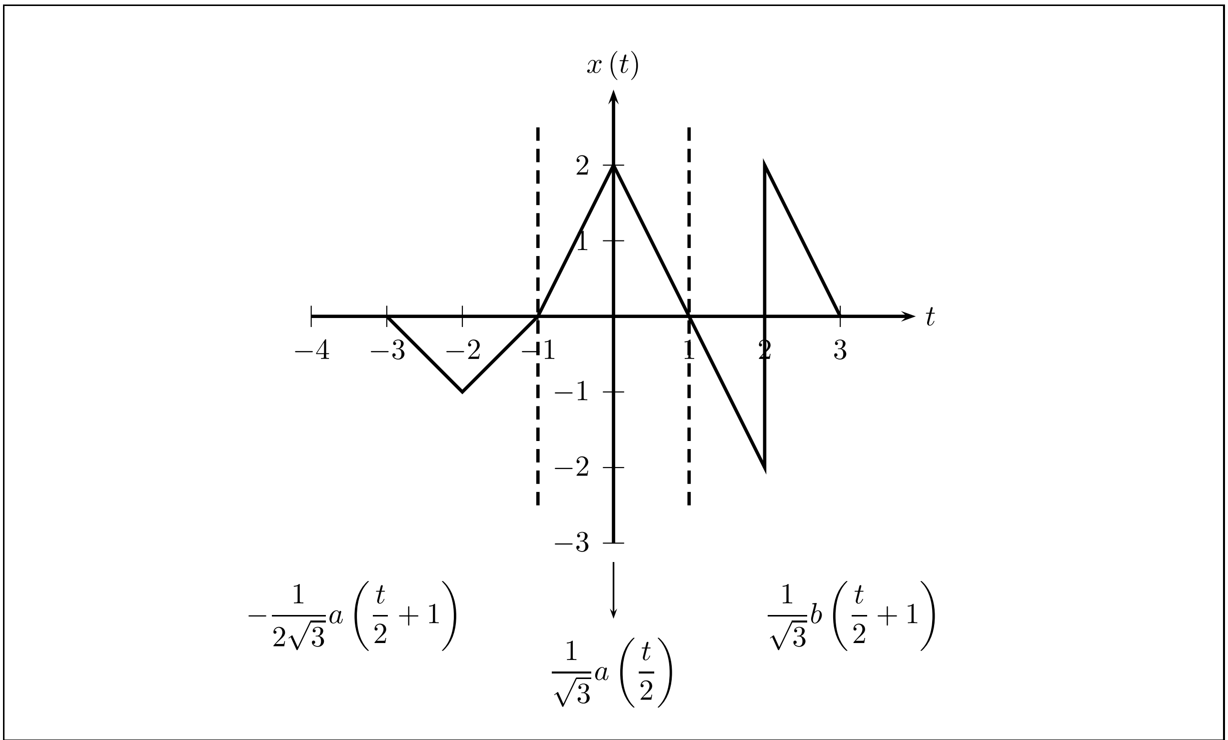
$$\begin{aligned} \int_{-\infty}^{\infty} a^2(t)\,dt &= \int_{-1/2}^{1/2} a^2(t)\,dt &&= 2\int_{0}^{1/2} a^2(t)\,dt \qquad (a^2(t) \text{ is even}) \\ &= 2\int_{0}^{1/2} 48\,(t - 1/2)^2\,dt &&= 2\cdot 48\,\frac{(t - 1/2)^3}{3}\Bigg|_{0}^{1/2} \\ &= 2\cdot 48\,\frac{1}{24} &&= 4 \end{aligned}$$

b) and c)

$$\int_{-\infty}^{\infty} b^2(t)\,dt = \int_{-\infty}^{\infty} a^2(t)\,dt = 4$$

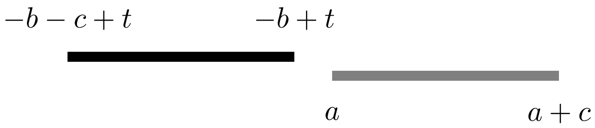
$$\int_{-\infty}^{\infty} a(t)\,b(t)\,dt = 0 \qquad (a(t)\,b(t) \text{ is odd})$$

d)



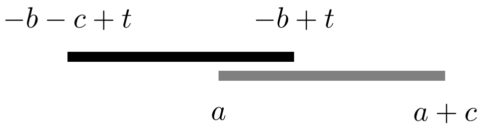
A2

A6) Consider that the signal $x(t)$ exists in the range $b < t < b + c$, and is zero outside of this region. And the signal $h(t)$ exists in the range $a < t < a + c$, and is zero outside of this region. Then the convolution integral between these two signals is computed for the following time intervals given in the following table.



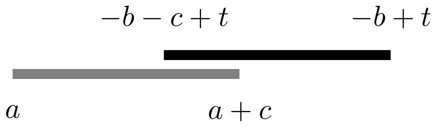
$t<a+b$

$$x(t)*h(t)=0$$



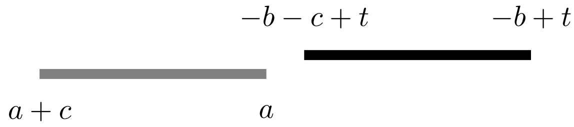
$a+b<t<a+b+c$

$$x(t)*h(t)=\int_{a}^{-b+t} h(\lambda)\,x(t-\lambda)\,d\lambda$$



$a+b+c<t<a+b+2c$

$$x(t)*h(t)=\int_{-b-c+t}^{a+c} h(\lambda)\,x(t-\lambda)\,d\lambda$$



$t<a+b+2c$

$$x(t)*h(t)=0$$

The output of the system in the question is computed by

$$y(t)=-2\,x(t)*h_1(t)*\delta(t-1)+x(t)*h_2(t)*u(t)$$

$x(t)*h_1(t)$

Range	Convolution
$-1/2>t,$	0
$-1/2<t<1/2,$	$\displaystyle\int_{0}^{1/2+t}(-\lambda+1)\cdot 1\,d\lambda=\left.-\frac{1}{2}\lambda^2+\lambda\right\vert_{0}^{1/2+t}=-\frac{1}{2}t^2+\frac{1}{2}t+\frac{3}{8}$
$1/2<t<3/2,$	$\displaystyle\int_{-1/2+t}^{1}(-\lambda+1)\cdot 1\,d\lambda=\left.-\frac{1}{2}\lambda^2+\lambda\right\vert_{-1/2+t}^{1}=\frac{1}{2}t^2-\frac{3}{2}t+\frac{9}{8}$
$1/2<t,$	0

$x(t) * h_1(t) * \delta(t-1)$ is one-unit delay of $x(t) * h_1(t)$

Range		$x(t) * h_1(t) * \delta(t-1)$
$-1/2 > (t-1)$,	$1/2 > t$,	0
$-1/2 < (t-1) < 1/2$,	$1/2 < t < 3/2$,	$-\frac{1}{2}(t-1)^2 + \frac{1}{2}(t-1) + \frac{3}{8}$
$1/2 < (t-1) < 3/2$,	$3/2 < t < 5/2$,	$\frac{1}{2}(t-1)^2 - \frac{3}{2}(t-1) + \frac{9}{8}$
$1/2 < (t-1)$,	$5/2 < t$,	0

$x(t) * h_2(t)$

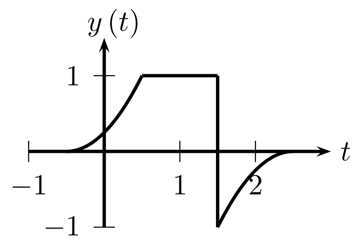
Range	Convolution
$-1/2 > t$,	0
$-1/2 < t < 1/2$,	$\int_{0}^{1/2+t} 2 \cdot 1\, d\lambda = 2\lambda \Big\vert_{0}^{1/2+t} = 2t+1$
$1/2 < t < 3/2$,	$\int_{-1/2+t}^{1} 2 \cdot 1\, d\lambda = 2\lambda \Big\vert_{-1/2+t}^{1} = 3-2t$
$1/2 < t$,	0

$$x(t) * h_2(t) * u(t) = \int_{-\infty}^{t} x(\lambda) * h_2(\lambda)\, d\lambda$$

Range	$x(t) * h_2(t) * u(t)$
$-1/2 > t,$	$\int_{-\infty}^{t} 0\, d\lambda = 0$
$-1/2 < t < 1/2,$	$\int_{-1/2}^{t} (2\lambda + 1)\, d\lambda = t^2 + t + \frac{1}{4}$
$1/2 < t < 3/2,$	$t^2 + t + \frac{1}{4}\Big\vert_{t=1/2} + \int_{1/2}^{t} (3 - 2\lambda)\, d\lambda = -t^2 + 3t - \frac{1}{4}$
$1/2 < t,$	$-t^2 + 3t - \frac{1}{4}\Big\vert_{t=1/2} = 2$

Finally, we are ready to write the output, $y(t)$.

$$y(t) = \begin{cases} t^2 + t + \frac{1}{4}, & -1/2 < t < 1/2 \\ 1, & 1/2 < t < 3/2, \\ -t^2 + 5t - \frac{25}{4}, & 3/2 < t < 5/2 \\ 0, & 5/2 < t \end{cases}$$



A3

A4) a)

$$\begin{aligned} Y(s) &= A(s)\left[X(s) + B(s)Y(s)\right] \\ Y(s) &= A(s)X(s) + A(s)B(s)Y(s) \\ Y(s)\left[1 - A(s)B(s)\right] &= A(s)X(s) \\ H(s) &= \frac{Y(s)}{X(s)} \\ &= \frac{A(s)}{1 - A(s)B(s)} \end{aligned}$$

b)

$$\begin{aligned}
1 - A(s)\,B(s) &= 1 - 8 \cdot \frac{1}{s^2+9} \\
&= \frac{s^2+1}{s^2+9} \\
s^2+1 &= 0 \\
s^2 &= -1 \\
s &= \pm j \qquad \text{(roots of } 1 - A(s)\,B(s) = 0\text{)}
\end{aligned}$$

$$\begin{aligned}
H(s) &= \frac{8s^2+72}{s^2+1} \\
&= \frac{8s^2+8+64}{s^2+1} \\
&= 8 + \frac{64}{s^2+1}
\end{aligned}$$

$$\begin{aligned}
h(t) &= \mathsf{L}^{-1}\left[\,H(s)\,\right] \\
&= 8\,\delta(t) + 64\sin(t)\,u(t)
\end{aligned}$$

c)

$$\begin{aligned}
H(s) &= \frac{-9}{1 - \frac{-9}{s+3}} = \frac{-9}{\frac{s+12}{s+3}} \\
&= \frac{-9s-27}{s+12} = \frac{-9s-27-81+81}{s+12} \\
&= -9 + \frac{81}{s+12}
\end{aligned}$$

$$h(t) = \mathsf{L}^{-1}\left[\,H(s)\,\right] = -9\,\delta(t) + 81e^{-12t}u(t)$$

d)

$$
\begin{aligned}
H(s) &= \frac{-2}{1-\dfrac{-2}{s+2}} = \frac{\dfrac{-2}{s+4}}{s+2} \\
&= \frac{-2s-4}{s+4} = \frac{-2s-4-4+4}{s+4} \\
&= -2+\frac{4}{s+4} \\
h(t) &= \mathsf{L}^{-1}\left[H(s)\right] = -2\,\delta(t)+4e^{-4t}u(t)
\end{aligned}
$$

A5) a) By employing the Kirchhoff's current law at the node joining the resistor, the inductor and the capacitor we obtain

$$
\frac{1}{\sqrt{2}}y'(t)+\frac{y(t)}{1}+\frac{1}{\sqrt{2}}\int_{-\infty}^{t}\left(y(\lambda)-x(\lambda)\right)d\lambda = 0
$$

Differentiating both sides and multiplying by $\sqrt{2}$, we have

$$
\begin{aligned}
y''(t)+\sqrt{2}\,y'(t)+y(t)-x(t) &= 0 \\
y''(t)+\sqrt{2}\,y'(t)+y(t) &= x(t)
\end{aligned}
$$

b)

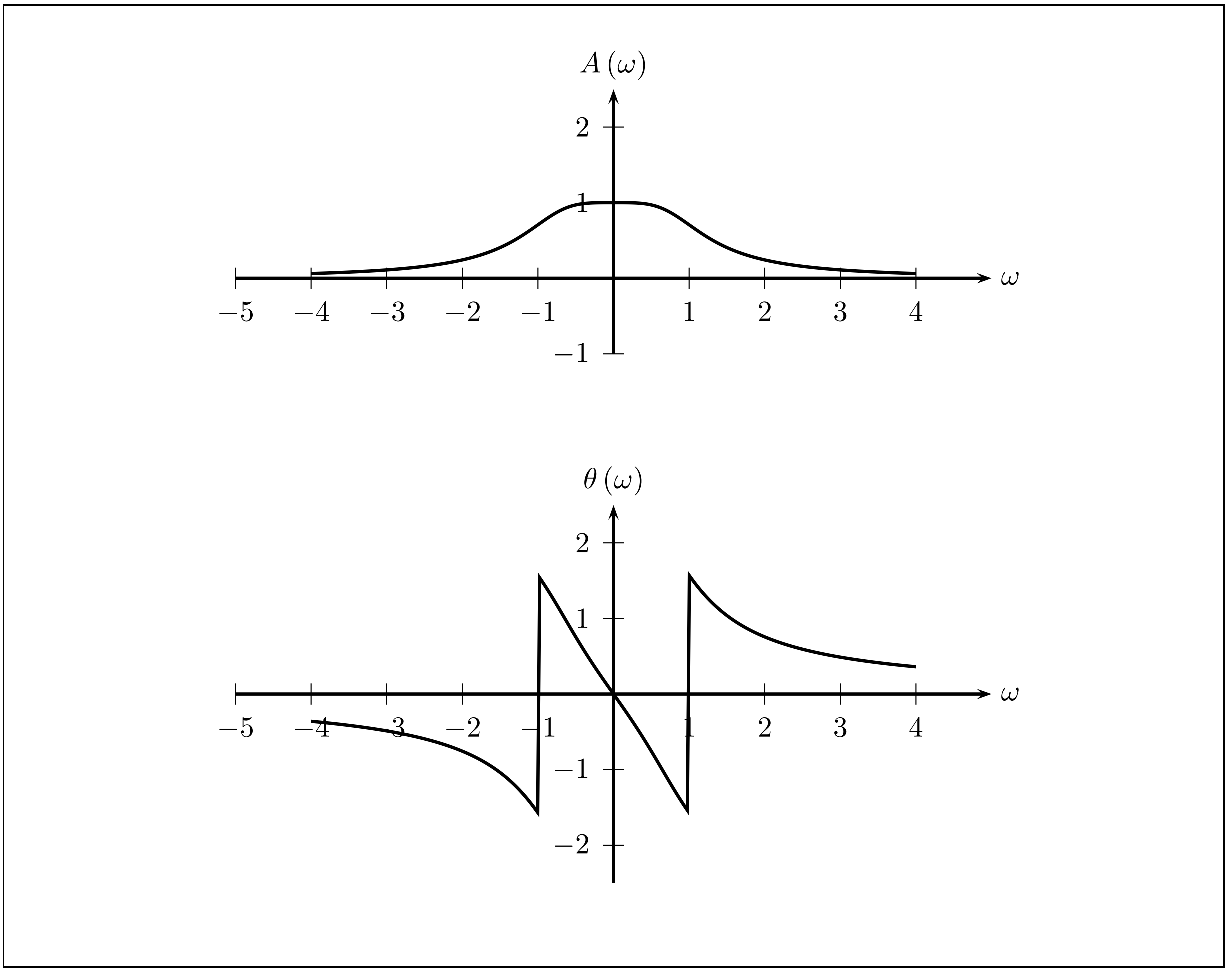
$$
\begin{aligned}
\mathsf{F}\left[y''(t)+\sqrt{2}\,y'(t)+y(t)\right] &= \mathsf{F}\left[x(t)\right] \\
(j\omega)^2\,Y(\omega)+\sqrt{2}\,j\omega Y(\omega)+Y(\omega) &= X(\omega)
\end{aligned}
$$

$$
\begin{aligned}
H(\omega) = \frac{Y(\omega)}{X(\omega)} &= \frac{1}{(j\omega)^2+\sqrt{2}\,j\omega+1} \\
&= \frac{1}{1-\omega^2+\sqrt{2}\,j\omega}
\end{aligned}
$$

c)

$$
\begin{aligned}
H(\omega)\,H^*(\omega) &= \frac{1}{1-\omega^2+\sqrt{2}\,j\omega}\,\frac{1}{1-\omega^2-\sqrt{2}\,j\omega} = \frac{1}{(1-\omega^2)^2+2\omega^2} \\
&= \frac{1}{1+\omega^4} \\
|H(\omega)| &= A(\omega) = \frac{1}{\sqrt{1+\omega^4}} \\
\sphericalangle H(\omega) &= \theta(\omega) = -\arctan\left(\frac{\sqrt{2}\omega}{1-\omega^2}\right)
\end{aligned}
$$

ω	$-\infty$	-1^-	-1^+	0	1^-	1^+	∞
$A(\omega)$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\theta(\omega)$	0	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	0



A5c

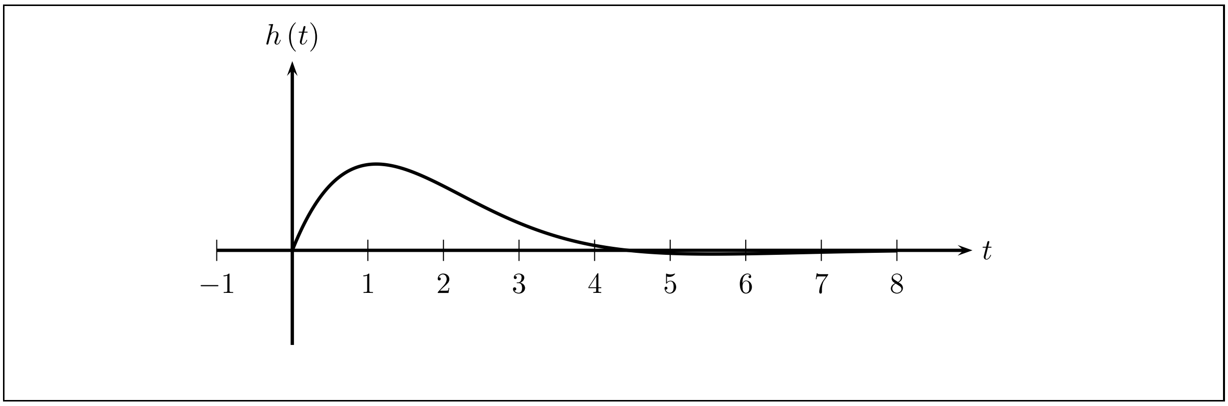
d)

$$
\begin{aligned}
(j\omega)^2+\sqrt{2}\,j\omega+1 &= \left(j\omega+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(j\omega+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)\\
\frac{1}{(j\omega)^2+\sqrt{2}\,j\omega+1} &= \frac{1}{\left(j\omega+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}\right)\left(j\omega+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}\right)}\\
&= \frac{A}{j\omega+\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}}}+\frac{B}{j\omega+\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}}}
\end{aligned}
$$

$$
\begin{aligned}
A &= \lim_{j\omega \to -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}} \left(j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right) \frac{1}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} \\
&= \lim_{j\omega \to -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}} \frac{1}{j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}} \\
&= j\frac{1}{\sqrt{2}} \\
B &= A^* \qquad \text{(because the coefficients of the rational polynomial are real)} \\
&= -j\frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
H(\omega) &= \frac{1}{(j\omega)^2 + \sqrt{2}\, j\omega + 1} \\
&= j\frac{1}{\sqrt{2}} \frac{1}{j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}} - j\frac{1}{\sqrt{2}} \frac{1}{j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}}
\end{aligned}
$$

$$
\begin{aligned}
h(t) &= \mathsf{F}\left[H(\omega)\right] \\
&= j\frac{1}{\sqrt{2}}\, e^{-\left(1/\sqrt{2} + j1/\sqrt{2}\right)t}\, u(t) - j\frac{1}{\sqrt{2}}\, e^{-\left(1/\sqrt{2} - j1/\sqrt{2}\right)t}\, u(t) \\
&= \frac{1}{\sqrt{2}}\, e^{-\left(1/\sqrt{2}\right)t}\, u(t) \left[j e^{-j\left(1/\sqrt{2}\right)t} - j e^{j\left(1/\sqrt{2}\right)t} \right] \\
&= \frac{1}{\sqrt{2}}\, e^{-\left(1/\sqrt{2}\right)t}\, u(t)\; 2\sin\left(\frac{1}{\sqrt{2}} t\right) \\
&= \sqrt{2}\, e^{-\left(1/\sqrt{2}\right)t} \sin\left(\frac{1}{\sqrt{2}} t\right) u(t)
\end{aligned}
$$



A5d