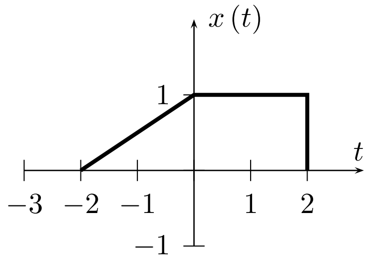
Instructions Answer all questions. Give your answers clearly. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Calculator and cell phone are not allowed in the exam. Each question is worth 15 points. **Time** 135 minutes.

1	2	3	4	5	6	7	8	9	10	TOTAL

QUESTIONS

Q1) For the signal given below plot $y(t) = x(-2t+6) - 2x(2t-2)$.



Q2) Find the impulse response of the following causal-LTI system. Do all computations in time domain.

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Q3) A causal-LTI system is given as in the following.

$$\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

The homogenous solution of the following causal-LTI system is

$$h(t) = Ae^{-2t}, \qquad t > 0$$

Since the highest derivative of the input and the output are equal it contains an impulse at the origin. Therefore the impulse response of this system is

$$h(t) = Ae^{-2t}u(t) + B\delta(t)$$

Find the unknown coefficients A, and B by replacing the impulse response in the differential equation. From the determined impulse response compute $h(0^+)$.

Note that,

$$[g(t)\delta(t)]' = g'(0)\delta(t) + g(t)\delta'(t)$$

Q4) The derivative and the integral are approximated as follows.

$$\frac{d}{dt}x(t) \approx \frac{x(t+\Delta) - x(t)}{\Delta}, \qquad 1 \gg \Delta > 0$$

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \approx \Delta\sum_{k=1}^{\infty} x(t - k\cdot\Delta)$$

Consider that the response of an LTI system for input $x(t)$ is $y(t)$. Employing the linearity and time-invariance properties find the output for the two inputs; a) $\frac{d}{dt}x(t)$, b) $\int\limits_{-\infty}^{t} x(\lambda)\, d\lambda$, in terms of $y(t)$.

Q5)
a) The impulse response of an LTI system is $h(t) = -4e^{-3t}u(t)$. Find the response of the system for input $\delta'(t) + 2\delta(t)$.

Note that,

$$[\,g(t)\,\delta(t)\,]' = g'(0)\,\delta(t) + g(t)\,\delta'(t)$$

Input	Output
$\delta(t)$	$h(t)$
$\delta'(t)$	$h'(t)$

b) The impulse response of an LTI system is $h[n] = -4\left(\frac{1}{3}\right)^n u[n]$. Find the response of the system for input $\delta[n-1] + 2\delta[n]$.

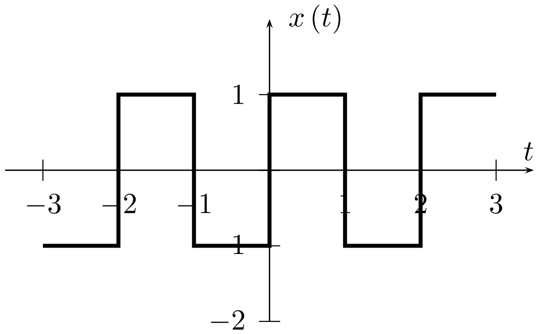
Q6) Find the impulse response of the following causal-LTI system. Do all computations in time domain.

$$y[n] - y[n-1] + \frac{2}{9}y[n-2] = x[n] + \frac{1}{2}x[n-1]$$

Q7) Find a) trigonometric Fourier series coefficients (cos, sin expansion), b) complex Fourier series coefficients of the following periodic signal.

$$x(t) = 10\cos\left(\frac{3\pi}{5}t\right) - 8\sin\left(\frac{4\pi}{5}t\right) + 3$$

Q8) Compute trigonometric Fourier series coefficients of the following periodic signal.



Q9) Check if the following signal is Fourier transformable. If yes, compute its Fourier transform by using the transform integral.

$$x(t) = e^{-t}\cos(2t)\,u(t)$$

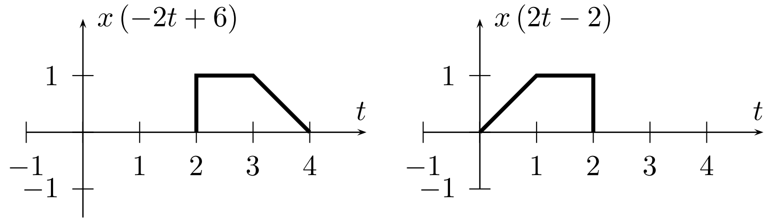
Q10)
a) Find frequency response of the following causal-LTI system by using Fourier transformation (do all computations in the frequency domain).

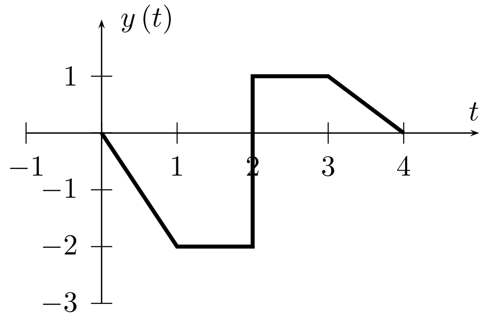
$$\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 9y(t) = 3\frac{d}{dt}x(t) + 6x(t)$$

b) Find the impulse response by computing inverse Fourier transform of the frequency response. Employ indirect method; partial fraction expansion to compute the inverse Fourier transform.

ANSWERS

A1)





A2) When the input $x(t) = \delta(t)$, the output $y(t) = h(t)$.

$$\mathcal{D}^2 h(t) + 5\mathcal{D}h(t) + 4h(t) = \mathcal{D}\delta(t) + 2\delta(t)$$

As the system is causal $h(t) = 0$ for $t < 0$, there is a non-zero solution for $t > 0$. Since the impulse disappears after $t = 0$, we have

$$\mathcal{D}^2 h(t) + 5\mathcal{D}h(t) + 4h(t) = 0, \qquad t > 0$$

The characteristic equation of this homogenous differential equation is

$$D^2 + 5D + 4 \quad = \quad (D+1)(D+4) \quad = \quad 0$$

The roots of the characteristic equation are $D_1 = -1$ and $D_2 = -4$. Therefore,
The unknown coefficients are extracted from the initial values $h\left(0^+\right)$ and $h'\left(0^+\right)$. To get the initial values the differential equation is integrated once and twice. And the finite integrals over interval $\left[\,0^-,\ 0^+\,\right]$ are computed.

$$\begin{array}{lcll}
\mathcal{D}^2 h(t) + 5\mathcal{D}h(t) + 4h(t) & = & \mathcal{D}\delta(t) + 2\delta(t) & \\
\mathcal{D}h(t) + 5h(t) + 4\mathcal{D}^{-1}h(t) & = & \delta(t) + 2u(t) & \text{integrated once} \\
h(t) + 5\mathcal{D}^{-1}h(t) + 4\mathcal{D}^{-2}h(t) & = & u(t) + 2tu(t) & \text{integrated twice}
\end{array}$$

$$\begin{array}{lcl}
h(t)\Big|_{0^-}^{0^+} + \underbrace{5\,\mathcal{D}^{-1}h(t)\Big|_{0^-}^{0^+}}_{\approx 0} + \underbrace{4\,\mathcal{D}^{-2}h(t)\Big|_{0^-}^{0^+}}_{\approx 0} & = & u(t)\Big|_{0^-}^{0^+} + \underbrace{2\,tu(t)\Big|_{0^-}^{0^+}}_{\approx 0} \\
h(0^+) - h(0^-) & = & u(0^+) - u(0^-) \\
h(0^+) & = & 1
\end{array}$$

$$
\begin{aligned}
\mathcal{D}h(t)\Big|_{0^-}^{0^+} + 5h(t)\Big|_{0^-}^{0^+} + \underbrace{4\,\mathcal{D}^{-1}h(t)\Big|_{0^-}^{0^+}}_{\approx 0} &= \delta(t)\Big|_{0^-}^{0^+} + 2u(t)\Big|_{0^-}^{0^+} \\
h'(0^+) - h'(0^-) + 5h(0^+) - 5h(0^-) &= \delta(0^+) - \delta(0^-) + 2u(0^+) - 2u(0^-) \\
h'(0^+) + 5 &= 2 \\
h'(0^+) &= -3
\end{aligned}
$$

Now, we are ready to compute the unknown coefficients.

$$
\begin{aligned}
h(t) &= Ae^{-t} + Be^{-4t}, && t > 0 \\
h'(t) &= -Ae^{-t} - 4Be^{-4t}, && t > 0
\end{aligned}
$$

$$
\begin{aligned}
h(0^+) &= A + B &&= 1 \\
h'(0^+) &= -A - 4B &&= -3
\end{aligned}
$$

$$
\begin{aligned}
-A - B - 3B &= -1 - 3B &&= -3 \\
B &= \frac{2}{3} \\
A &= 1 - B &&= \frac{1}{3}
\end{aligned}
$$

Finally, we get the impulse response.

$$
h(t) = \frac{1}{3}e^{-t}u(t) + \frac{1}{3}e^{-4t}u(t)
$$

Alternatively, the unknown coefficients can be found by replacing $h(t)$ in the differential equation.

$$
\begin{aligned}
h(t) &= Ae^{-t}u(t) + Be^{-4t}u(t) \\
h'(t) &= -Ae^{-t}u(t) - 4Be^{-4t}u(t) + A\delta(t) + B\delta(t) \\
h''(t) &= Ae^{-t} + 16Be^{-4t} - A\delta(t) - 4B\delta(t) + A\delta'(t) + B'\delta(t)
\end{aligned}
$$

$$
\begin{aligned}
h''(t) + 5h'(t) + 4h(t) &= [\,-A\delta(t) - 4B\delta(t) + A\delta'(t) + B'\delta(t)\,] + 5\,[\,A\delta(t) + B\delta(t)\,] \\
(4A + B)\,\delta(t) + (A + B)\,\delta'(t) &= \delta'(t) + 2\delta(t)
\end{aligned}
$$

The equality holds in the last equation if the coefficients of the $\delta(t)$ and $\delta'(t)$ in the both sides are equal.

$$
\begin{aligned}
4A + B &= 2 \\
A + B &= 1
\end{aligned}
$$

$$
\begin{aligned}
3A + A + B &= 3A + 1 &&= 2 \\
A &= \frac{1}{3} \\
B &= 1 - A &&= \frac{2}{3}
\end{aligned}
$$

A3) The impulse response of the following causal-LTI system;

$$\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + x(t)$$

is already given as

$$h(t) = Ae^{-2t}u(t) + B\delta(t)$$

To identify the unknown coefficients this solution is replaced in the differential equation;

$$h'(t) + 2h(t) = 2\delta'(t) + \delta(t)$$

$$\begin{aligned} h(t) &= Ae^{-2t}u(t) + B\delta(t) \\ h'(t) &= -2Ae^{-2t}u(t) + A\delta(t) + B\delta'(t) \end{aligned}$$

The exponential terms vanish in the equation since $Ae^{-2t}u(t)$ is the homogenous solution. Therefore, there is nothing to do with the exponentials. Simply, it is enough to replace $h(t)$ with $B\delta(t)$ and $h'(t)$ with $A\delta(t) + B\delta'(t)$ in the differential equation.

$$\begin{aligned} h'(t) + 2h(t) &= A\delta(t) + B\delta'(t) + 2B\delta(t) \\ (A+2B)\delta(t) + B'\delta(t) &= \delta'(t) + 2\delta(t) \end{aligned}$$

By equating the coefficients of $\delta(t)$ and $\delta'(t)$ in the both sides of the last equation we have

$$\begin{aligned} A + 2B &= 1 \\ B &= 2 \end{aligned}$$

The coefficients are then,

$$\begin{aligned} A &= -3 \\ B &= 2 \end{aligned}$$

Consequently,

$$h(t) = -3e^{-2t}u(t) + 2\delta(t)$$

And the value at $t = 0^+$

$$\begin{aligned} h(0^+) &= -3e^{-2\cdot 0^+}u(0^+) + 2\delta(0^+) \\ &= -3 \cdot 1 + 2 \cdot 0 \\ &= -3\,. \end{aligned}$$

A4) The output signal is related to the input via a function characterizing the system.

$$T(x(t)) = y(t)$$

a) Response of the system for the input; $\frac{d}{dt}x(t)$.

$$\begin{aligned}
T\left(\frac{d}{dt}x(t)\right) &\approx T\left(\frac{x(t+\Delta)-x(t)}{\Delta}\right) \\
&= \frac{1}{\Delta}T\left(x(t+\Delta)-x(t)\right) && \text{by the homogeneity property} \\
&= \frac{1}{\Delta}T\left(x(t+\Delta)\right)-\frac{1}{\Delta}T\left(x(t)\right) && \text{by the additivity property} \\
&= \frac{1}{\Delta}y(t+\Delta)-\frac{1}{\Delta}y(t) && \text{by the time-invariance property} \\
&\approx \frac{d}{dt}y(t) \qquad \blacksquare
\end{aligned}$$

b) Response of the system for the input; $\int\limits_{-\infty}^{t} x(\lambda)\,d\lambda$.

$$\begin{aligned}
T\left(\int\limits_{-\infty}^{t} x(\lambda)\,d\lambda\right) &\approx T\left(\Delta\sum_{k=1}^{\infty}x(t-k\cdot\Delta)\right) \\
&= \Delta\,T\left(\sum_{k=1}^{\infty}x(t-k\cdot\Delta)\right) && \text{by the homogeneity property} \\
&= \Delta\sum_{k=1}^{\infty}T\left(x(t-k\cdot\Delta)\right) && \text{by the additivity property} \\
&= \Delta\sum_{k=1}^{\infty}y(t-k\cdot\Delta) && \text{by the time-invariance property} \\
&\approx \int\limits_{-\infty}^{t} y(\lambda)\,d\lambda \qquad \blacksquare
\end{aligned}$$

A4) The linearity and time-invariance properties of the system are employed to compute the output.
a)

$$\begin{aligned}
T\left(\delta'(t)+2\delta(t)\right) &= T\left(\delta'(t)\right)+2T\left(\delta(t)\right) = h'(t)+2h(t) \\
h'(t) &= 12e^{-3t}u(t)-4\delta(t) \\
h'(t)+2h(t) &= 4e^{-3t}u(t)-4\delta(t) \\
T\left(\delta'(t)+2\delta(t)\right) &= 4e^{-3t}u(t)-4\delta(t)
\end{aligned}$$

b)

$$\begin{aligned}
T\left(\delta[n-1]+2\delta[n]\right) &= T\left(\delta[n-1]\right)+2T\left(\delta[n]\right) &&= h[n-1]+2h[n] \\
h[n-1] &= -4\left(\frac{1}{3}\right)^{n-1}u[n-1] &&= -12\left(\frac{1}{3}\right)^{n}u[n-1] \\
&= -12\left(\frac{1}{3}\right)^{n}\left(u[n]-\delta[n]\right) &&= -12\left(\frac{1}{3}\right)^{n}u[n]+12\delta[n] \\
h[n-1]+2h[n] &= -20\left(\frac{1}{3}\right)^{n}u[n]+12\delta[n] \\
T\left(h[n-1]+2h[n]\right) &= -20\left(\frac{1}{3}\right)^{n}u[n]+12\delta[n]
\end{aligned}$$

A6) The response for the unit impulse is $h[n]$. Therefore, the difference equation for the impulse response becomes as in the following.

$$h[n]-h[n-1]+\frac{2}{9}h[n-2]=\delta[n]+\frac{1}{2}\delta[n-1]$$

Since the system is causal $h[n]=0$ for $n<0$. The difference equation has a non-zero solution only for $n\geq 0$. While $n>1$, right side of the difference equation is zero and we have a homogenous difference equation.

$$h[n]-h[n-1]+\frac{2}{9}h[n-2]=0, \qquad n>1$$

The characteristics equation of this homogenous equation is

$$1-D^{-1}+\frac{2}{9}D^{-2} \quad = \quad \left(1-\frac{1}{3}D^{-1}\right)\left(1-\frac{2}{3}D^{-1}\right) \quad = \quad 0\,.$$

and the roots of the characteristic equation are $D_1=\dfrac{1}{3}$ and $D_2=\dfrac{2}{3}$. Hence, the solution is

$$h[n]=A\left(\frac{1}{3}\right)^{n}+B\left(\frac{2}{3}\right)^{n}, \qquad n>1$$

The unknown coefficients can be computed from the initial values; $h[0]$ and $h[1]$. The initial values can be obtained directly from the difference equation;

$$\begin{aligned}
n \;=\; 0;\quad h[0]-h[-1]+\frac{2}{9}h[-2] &= \delta[0]+\frac{1}{2}\delta[-1] \\
h[0]-0+\frac{2}{9}\cdot 0 &= 1+\frac{1}{2}\cdot 0 \\
h[0] &= 1
\end{aligned}$$

$$\begin{aligned}
n \;=\; 1;\quad h[1]-h[0]+\frac{2}{9}h[-1] &= \delta[1]+\frac{1}{2}\delta[0] \\
h[1]-1+\frac{2}{9}\cdot 0 &= 0+\frac{1}{2}\cdot 1 \\
h[1] &= \frac{1}{2}+1 \\
&= \frac{3}{2}
\end{aligned}$$

Now, the coefficients can be computed.

$$\begin{aligned} h[0] &= A + B &= 1 \\ h[1] &= A\frac{1}{3} + B\frac{2}{3} &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} A\frac{1}{3} + B\frac{2}{3} &= \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}B \\ &= \frac{1}{3} + \frac{1}{3}B &= \frac{3}{2} \\ \frac{1}{3}B &= \frac{3}{2} - \frac{1}{3} &= \frac{7}{6} \\ B &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} A &= 1 - B = 1 - \frac{7}{2} \\ &= -\frac{5}{2} \end{aligned}$$

Finally, the solution is obtained.

$$h[n] = \begin{cases} 0, & n < 0 \\ -\frac{5}{2}\left(\frac{1}{3}\right)^n + \frac{7}{2}\left(\frac{2}{3}\right)^n, & n \geq 0 \end{cases} = -\frac{5}{2}\left(\frac{1}{3}\right)^n u[n] + \frac{7}{2}\left(\frac{2}{3}\right)^n u[n]$$

Alternatively, the unknown coefficients can be obtained by replacing the impulse response in the difference equation. The time shifted versions of the unit step function are replaced by their equivalents.

$$\begin{aligned} u[n-1] &= u[n] - \delta[n] \\ u[n-2] &= u[n] - \delta[n] - \delta[n-1] \end{aligned}$$

Now compute $h[n-1]$, and $h[n-2]$.

$$\begin{aligned} h[n] &= A\left(\frac{1}{3}\right)^n u[n] + B\left(\frac{2}{3}\right)^n u[n] \\ h[n-1] &= A\left(\frac{1}{3}\right)^{n-1} u[n-1] + B\left(\frac{2}{3}\right)^{n-1} u[n-1] \\ &= 3A\left(\frac{1}{3}\right)^n (u[n] - \delta[n]) + \frac{3}{2}B\left(\frac{2}{3}\right)^n (u[n] - \delta[n]) \\ &= 3A\left(\frac{1}{3}\right)^n u[n] + \frac{3}{2}B\left(\frac{2}{3}\right)^n u[n] - 3A\delta[n] - \frac{3}{2}B\delta[n] \\ h[n-2] &= 3A\left(\frac{1}{3}\right)^{n-1} u[n-1] + \frac{3}{2}B\left(\frac{2}{3}\right)^{n-1} u[n-1] - 3A\delta[n-1] - \frac{3}{2}B\delta[n-1] \\ &= 9A\left(\frac{1}{3}\right)^n u[n] + \frac{9}{4}B\left(\frac{2}{3}\right)^n u[n] - 9A\delta[n] - \frac{9}{4}B\delta[n] - 3A\delta[n-1] - \frac{3}{2}B\delta[n-1] \end{aligned}$$

The exponentials will be eliminated since they are homogenous solutions of the difference equations.

Therefore only impulse terms will remain.

$$\begin{aligned}
h[n] - h[n-1] + \frac{2}{9}h[n-2] &= \\
&0 - \left(-3A\delta[n] - \frac{3}{2}B\delta[n]\right) + \frac{2}{9}\left(-9A\delta[n] - \frac{9}{4}B\delta[n] - 3A\delta[n-1] - \frac{3}{2}B\delta[n-1]\right) \\
&= (A+B)\,\delta[n] + \left(-\frac{2}{3}A - \frac{1}{3}B\right)\delta[n-1] = \delta[n] + \frac{1}{2}\delta[n-1]
\end{aligned}$$

The equality is hold when

$$\begin{aligned}
A + B &= 1 \\
-\frac{2}{3}A - \frac{1}{3}B &= \frac{1}{2}
\end{aligned}$$

The coefficients are easily computed.

$$\begin{aligned}
-\frac{2}{3}A - \frac{1}{3}B &= -\frac{1}{3}A - \frac{1}{3}A - \frac{1}{3}B \\
&= -\frac{1}{3} - \frac{1}{3}A && = \frac{1}{2} \\
\frac{1}{3}A &= \frac{1}{2} + \frac{1}{3} && = \frac{5}{6} \\
A &= -\frac{5}{2}
\end{aligned}$$

$$\begin{aligned}
B &= 1 - A = 1 + \frac{5}{2} \\
&= \frac{7}{2}
\end{aligned}$$

A7) First we need to find the period of the signal. The signal is composed of two trigonometric functions and a constant. The period of cosine is

$$\begin{aligned}
\frac{3\pi}{5} &= \frac{2\pi}{T_1} \\
T_1 &= \frac{10}{3}
\end{aligned}$$

and the period of sine is

$$\begin{aligned}
\frac{4\pi}{5} &= \frac{2\pi}{T_2} \\
T_2 &= \frac{5}{2}
\end{aligned}$$

Consider that T is the fundamental period of $x(t)$. The fundamental periods of cosine and sine should repeat integer time in T seconds. Therefore,

$$nT_1 = mT_2 = T, \quad \text{for some positive integers, } n \text{ and } m$$

The smallest integers that satisfy the requirement are $n = 3$ and $m = 4$. Then the fundamental period of $x(t)$ is $T = 10$ seconds.
a) The trigonometric Fourier series is as in the following.

$$x(t) = a_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{2\pi}{T}kt\right) + \sum_{k=1}^{\infty} D_k \sin\left(\frac{2\pi}{T}kt\right)$$

The harmonic indices are needed.

$$\begin{aligned}\frac{3\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = 3\\ \frac{4\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = 4\end{aligned}$$

Therefore the non-zero series coefficients for this signal are

$$a_0 = 3 \qquad C_3 = 10 \qquad D_4 = -8$$

b) The complex Fourier series is as in the following.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

The cosine and sine are replaced by their complex exponential forms.

$$\begin{aligned}\cos\left(\frac{3\pi}{5}\right) &= \frac{1}{2}e^{j\frac{3\pi}{5}t} + \frac{1}{2}e^{-j\frac{3\pi}{5}t}\\ \sin\left(\frac{4\pi}{5}\right) &= \frac{1}{2j}e^{j\frac{4\pi}{5}t} - \frac{1}{2j}e^{-j\frac{4\pi}{5}t}\end{aligned}$$

$$\begin{aligned}x(t) &= 10\cos\left(\frac{3\pi}{5}t\right) - 8\sin\left(\frac{4\pi}{5}t\right) + 3\\ x(t) &= 10\left(\frac{1}{2}e^{j\frac{3\pi}{5}t} + \frac{1}{2}e^{-j\frac{3\pi}{5}t}\right) - 8\left(\frac{1}{2j}e^{j\frac{4\pi}{5}t} - \frac{1}{2j}e^{-j\frac{4\pi}{5}t}\right) + 3\\ &= -4je^{-j\frac{4\pi}{5}t} + 5e^{-j\frac{3\pi}{5}t} + 3 + 5e^{j\frac{3\pi}{5}t} + 4je^{j\frac{4\pi}{5}t}\end{aligned}$$

The series indices should be extracted.

$$\begin{aligned}-\frac{4\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = -4\\ -\frac{3\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = -3\\ 0 &= \frac{2\pi}{10}k \Rightarrow k = 0\\ \frac{3\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = 3\\ \frac{4\pi}{5} &= \frac{2\pi}{10}k \Rightarrow k = 4\end{aligned}$$

Therefore the non-zero series coefficients for this signal are

$$a_0 = 3 \qquad a_{-4} = -4j \qquad a_{-3} = 5$$

$$a_3 = 5 \qquad a_4 = 4j$$

A8) The periodic signal is odd; $x(-t) = -x(t)$. Hence, the trigonometric series does not contain a constant and a cosine term; the constant and the coefficients of the cosine term are zero; $a_0 = 0$ and $C_k = 0$. We should compute only coefficients of the sine terms. As it is observed from the figure the fundamental period is $T = 2$ seconds.

$$
\begin{aligned}
D_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2\pi}{T}kt\right)dt \\
&= \frac{2}{2}\int_{-1}^{1} x(t)\sin\left(\frac{2\pi}{2}kt\right)dt \\
&= \int_{-1}^{0}(-1)\sin(\pi kt)\,dt + \int_{0}^{1}(1)\sin(\pi kt)\,dt \\
&= -\int_{-1}^{0}\sin(\pi kt)\,dt + \int_{0}^{1}\sin(\pi kt)\,dt \\
&= \underbrace{-\int_{-1}^{0}\sin(\pi kt)\,dt}_{\int_{0}^{1}\sin(\pi kt)\,dt} + \int_{0}^{1}\sin(\pi kt)\,dt \\
&= 2\int_{0}^{1}\sin(\pi kt)\,dt \qquad = -\frac{2}{\pi k}\cos(\pi kt)\Big|_{0}^{1} \\
&= \frac{2}{\pi k}\left(1-\cos(\pi k)\right) \qquad = \frac{2}{\pi k}\left(1-(-1)^k\right)
\end{aligned}
$$

The even terms of the coefficients are zero.

$$
D_{2\ell} = 0 \qquad D_{2\ell-1} = \frac{4}{\pi(2\ell-1)}
$$

Therefore the series representation of the periodic signal is as follows.

$$
x(t) = \frac{4}{\pi}\sum_{\ell=1}^{\infty}\frac{1}{2\ell-1}\sin(\pi kt)
$$

A9) It is known that Fourier transform of a signal exists if it is summable. Therefore we should compute the integral of the absolute value of the signal. If this integral exists and is finite then we ensure that Fourier transform exists.

$$
\begin{aligned}
\int_{-\infty}^{\infty}|x(t)|\,dt &= \int_{-\infty}^{\infty}\left|e^{-t}\cos(2t)\,u(t)\right|dt &&= \int_{-\infty}^{\infty}e^{-t}\left|\cos(2t)\right|u(t)\,dt \\
&= \int_{0}^{\infty}e^{-t}\left|\cos(2t)\right|dt &&\leq \int_{0}^{\infty}e^{-t}dt \\
&= -e^{-t}\Big|_{0}^{\infty} &&= 1
\end{aligned}
$$

Hence we are certain that Fourier transform of the signal exists. Now, we compute its Fourier transform.

$$
e^{-t}\cos(2t) = e^{-t}\left(\frac{1}{2}e^{j2t}+\frac{1}{2}e^{-j2t}\right) = \frac{1}{2}e^{-(1-j2)t}+\frac{1}{2}e^{-(1+j2)t}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt &= \int_{-\infty}^{\infty} e^{-j\omega t} e^{-t} \cos(2t)\, u(t)\, dt \\
&= \int_{0}^{\infty} e^{-j\omega t} e^{-t} \cos(2t)\, dt \\
&= \frac{1}{2}\int_{0}^{\infty} e^{-j\omega t} e^{-(1-j2)t} dt + \frac{1}{2}\int_{0}^{\infty} e^{-j\omega t} e^{-(1+j2)t} dt \\
&= \frac{1}{2}\int_{0}^{\infty} e^{-(1-j2+j\omega)t} dt + \frac{1}{2}\int_{0}^{\infty} e^{-(1+j2+j\omega)t} dt
\end{aligned}
$$

$$
\begin{aligned}
\int_{0}^{\infty} e^{-(1-j2+j\omega)t} dt &= \frac{1}{-(1-j2+j\omega)} e^{-(1-j2+j\omega)t} \Big|_{0}^{\infty} \\
&= -\frac{1}{1-j2+j\omega} \left[e^{-(1-j2+j\omega)\cdot\infty} - 1 \right] \\
&= -\frac{1}{1-j2+j\omega} \left[e^{-\infty} e^{-j(-2+\omega)\cdot\infty} - 1 \right] \\
&= -\frac{1}{1-j2+j\omega} \left[0 \cdot e^{-j(-2+\omega)\cdot\infty} - 1 \right] \qquad (\ e^{-\infty} \to 0\ ) \\
&= -\frac{1}{1-j2+j\omega} \left[0 - 1 \right] \\
&= \frac{1}{1-j2+j\omega}
\end{aligned}
$$

Similarly,

$$
\int_{0}^{\infty} e^{-(1+j2+j\omega)t} dt = \frac{1}{1+j2+j\omega}
$$

Replacing these integrals we get the result.

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-j\omega t} e^{-t} \cos(2t)\, u(t)\, dt &= \frac{1}{2}\int_{0}^{\infty} e^{-(1-j2+j\omega)t} dt + \frac{1}{2}\int_{0}^{\infty} e^{-(1+j2+j\omega)t} dt \\
&= \frac{1}{2}\frac{1}{1-j2+j\omega} + \frac{1}{2}\frac{1}{1+j2+j\omega} \\
&= \frac{1+j\omega}{5+2j\omega+(j\omega)^2} \qquad \blacksquare
\end{aligned}
$$

A10) Fourier transform of both side of the differential equation yields

$$(j\omega)^2 Y(\omega) + 6j\omega Y(\omega) + 9Y(\omega) = 3j\omega X(\omega) + 6X(\omega)$$

The frequency response is the ratio of $Y(\omega)$ to $X(\omega)$ and can be extracted from the above equation.

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{3j\omega + 6}{(j\omega)^2 + 6j\omega + 9}$$

The partial fraction expansion of this frequency response is easy.

$$\begin{aligned}
H(\omega) &= \frac{3j\omega + 6}{(j\omega)^2 + 6j\omega + 9} &&= \frac{3(j\omega+2)}{(j\omega+3)^2} \\
&= \frac{3(j\omega+3) - 3}{(j\omega+3)^2} &&= \frac{3(j\omega+3)}{(j\omega+3)^2} + \frac{-3}{(j\omega+3)^2} \\
&= \frac{3}{j\omega+3} - \frac{3}{(j\omega+3)^2}
\end{aligned}$$

From the table of Fourier transforms we know that

$$(j\omega)^2 Y(\omega) + 6j\omega Y(\omega) + 9Y(\omega) = 3j\omega X(\omega) + 6X(\omega)$$

$$\begin{aligned}
\mathcal{F}^{-1}\left[\frac{1}{j\omega+3}\right] &= e^{-3t}u(t) \\
\mathcal{F}^{-1}\left[\frac{1}{(j\omega+3)^2}\right] &= te^{-3t}u(t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
h(t) = \mathcal{F}^{-1}[H(\omega)] &= 3\mathcal{F}^{-1}\left[\frac{1}{j\omega+3}\right] - 3\mathcal{F}^{-1}\left[\frac{1}{(j\omega+3)^2}\right] \\
&= 3e^{-3t}u(t) - 3te^{-3t}u(t)
\end{aligned}$$