Instructions Answer all questions. Give your answers clearly. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Calculator and cell phone are not allowed in the exam. Each question is worth 12.5 points. Time 120 minutes.


## QUESTIONS

Q1) Block diagram of an LTI and causal system is given as in the following. Extract inputoutput relation of the system (in time domain) from its block diagram implementation.


Q2) Find impulse response of the following LTI-causal system.

$$
y[n]-\frac{5}{6} y[n-1]+\frac{1}{6} y[n-2]=x[n]+\frac{1}{2} x[n-1]
$$

Q3 In the following circuit, $e(t)$ is input and $v(t)$ is the output. Find impulse response of the circuit.


Q4) Compute convolution sum; $x[n] * y[-n]$ for the following signals.


Q5) Impulse response; $h(t)$, of an LTI system is given as in the following figure. Find and plot the output of this system form the input; $x(t)$, given below.



Q6) a) An auxiliary function is given as in the following. You can easily deduce that unit impulse function can be derived from this function by $\delta(t)=\lim _{\Delta \rightarrow 0} a(t)$. Using this auxiliary function compute $\int_{-\infty}^{\infty} \delta^{\prime}(t) d t$.

b) Using $\int_{-\infty}^{\infty} \frac{d}{d t}[\delta(t) g(t)] d t=\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t+\int_{-\infty}^{\infty} \delta(t) g^{\prime}(t) d t$ compute $\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t$. Here, $g(t)$ is a smooth function (the function and its derivatives are continuous).

Q7) Plot direct form II type block diagram realization of the system described by the following differential equation.

$$
y^{\prime \prime}(t)+5 y^{\prime}(t)+6 y(t)=-x^{\prime}(t)+2 x(t)
$$

Q8) Find impulse response of the following LTI-causal system.

$$
y^{\prime \prime}(t)+5 y^{\prime}(t)+6 y(t)=-x^{\prime}(t)+2 x(t)
$$

Q9 In the following circuit, $x(t)$ is input and $y(t)$ is the output. Check both additivity and homogeneity of the system. Take two signals; $x_{1}(t)=1$ and $x_{2}(t)=2$, and $\alpha=2$ for the additivity and homogeneity testings.


Q10 A continuous-time signal; $x(t)$, is given in the following. Find and plot $y(t)=x(-2 t+2)-$ $x(2 t-10)$


## ANSWERS

A1)

$$
\begin{aligned}
v[n] & =x[n]+\frac{5}{6} \mathcal{D}^{-1} v[n]-\frac{1}{6} \mathcal{D}^{-2} v[n] \\
y[n] & =v[n]+\frac{1}{2} \mathcal{D}^{-1} v[n]
\end{aligned}
$$

$$
v[n]=\frac{1}{1-\frac{5}{6} \mathcal{D}^{-1}+\frac{1}{6} \mathcal{D}^{-2}} x[n]
$$

$$
y[n]=\left(1+\frac{1}{2} \mathcal{D}^{-1}\right) \frac{1}{1-\frac{5}{6} \mathcal{D}^{-1}+\frac{1}{6} \mathcal{D}^{-2}} x[n]
$$

$$
\left(1-\frac{5}{6} \mathcal{D}^{-1}+\frac{1}{6} \mathcal{D}^{-2}\right) y[n] \quad=\left(1+\frac{1}{2} \mathcal{D}^{-1}\right) x[n]
$$

$$
y[n]-\frac{5}{6} y[n-1]+\frac{1}{6} y[n-2]=x[n]+\frac{1}{2} x[n-1]
$$

A2) First we compute initials

$$
\begin{array}{rlr}
h[n] & -\frac{5}{6} h[n-1]+\frac{1}{6} h[n-2]=\delta[n]+\frac{1}{2} \delta[n-1] \\
h[n] & =\frac{5}{6} h[n-1]-\frac{1}{6} h[n-2]+\delta[n]+\frac{1}{2} \delta[n-1] & \\
h[0] & =\frac{5}{6} h[-1]-\frac{1}{6} h[-2]+\delta[0]+\frac{1}{2} \delta[-1] & = \\
& =\frac{5}{6} \cdot 0-\frac{1}{6} \cdot 0+1+\frac{1}{2} \cdot 0 & =\frac{4}{3}
\end{array}
$$

The system is causal $(h[n]=0, \quad n<0)$ so we need to compute response of the system for only $n \geq 0$. The difference equation becomes a homogenous difference equation for $n \geq 2$.

$$
h[n]-\frac{5}{6} h[n-1]+\frac{1}{6} h[n-2]=0, \quad n \geq 2
$$

The characteristic equation of this homogenous equation is as follows

$$
\begin{aligned}
& 1-\frac{5}{6} \alpha^{-1}+\frac{1}{6} \alpha^{-2}=\left(1-\frac{1}{3} \alpha^{-1}\right)\left(1-\frac{1}{2} \alpha^{-1}\right)=0 \\
& \alpha=\frac{1}{3}, \quad \text { and } \quad \alpha=\frac{1}{2}
\end{aligned}
$$

are roots of the characteristic equation. Therefore the impulse response, for $n \geq 0$, is

$$
h[n]=c_{1}\left(\frac{1}{3}\right)^{n}+c_{2}\left(\frac{1}{2}\right)^{n}
$$

Using the initial values we can extract unknown coefficients $c_{1}$ and $c_{2}$.

$$
\begin{aligned}
h[0] & =c_{1}+c_{2}=1 \\
h[1] & =c_{1} \frac{1}{3}+c_{2} \frac{1}{2}=\frac{4}{3} \\
& =2 c_{1}+3 c_{2}=8
\end{aligned}
$$

Solving the above equations we get $c_{1}=-5$ and $c_{2}=6$. Consequently, the impulse response of this system is

$$
h[n]=-5\left(\frac{1}{3}\right)^{n} u[n]+6\left(\frac{1}{2}\right)^{n} u[n]
$$

## Alternative solution

Consider the following system

$$
y[n]-\frac{5}{6} y[n-1]+\frac{1}{6} y[n-2]=x[n]
$$

If the impulse response of this system is $g[n]$ then we write

$$
g[n]-\frac{5}{6} g[n-1]+\frac{1}{6} g[n-2]=\delta[n]
$$

The solution of the above difference equation is

$$
g[n]=-2\left(\frac{1}{3}\right)^{n} u[n]+3\left(\frac{1}{2}\right)^{n} u[n]
$$

By employing the linearity and time-invariance property we can write $h[n]$ in terms of $g[n]$.

$$
h[n]=g[n]+\frac{1}{2} g[n-1]
$$

Therefore

$$
\begin{aligned}
& h[n]=-2\left(\frac{1}{3}\right)^{n} u[n]+3\left(\frac{1}{2}\right)^{n} u[n]+\frac{1}{2}\left(-2\left(\frac{1}{3}\right)^{n-1} u[n-1]+3\left(\frac{1}{2}\right)^{n-1} u[n-1]\right) \\
& h[n]=-2\left(\frac{1}{3}\right)^{n} u[n]+3\left(\frac{1}{2}\right)^{n} u[n]-3\left(\frac{1}{3}\right)^{n} u[n-1]+3\left(\frac{1}{2}\right)^{n} u[n-1]
\end{aligned}
$$

Use $u[n-1]=u[n]-\delta[n]$ in the equation.

$$
\begin{aligned}
h[n]= & -2\left(\frac{1}{3}\right)^{n} u[n]+3\left(\frac{1}{2}\right)^{n} u[n]-3\left(\frac{1}{3}\right)^{n} u[n]+3\left(\frac{1}{2}\right)^{n} u[n] \\
& +3\left(\frac{1}{3}\right)^{n} \delta[n]-3\left(\frac{1}{2}\right)^{n} \delta[n] \\
= & -5\left(\frac{1}{3}\right)^{n} u[n]+6\left(\frac{1}{2}\right)^{n} u[n]+3 \delta[n]-3 \delta[n] \\
= & -5\left(\frac{1}{3}\right)^{n} u[n]+6\left(\frac{1}{2}\right)^{n} u[n]
\end{aligned}
$$

We finally get the result.
A3) The current on the capacitor is $0.5 \frac{d}{d t} v(t)$. The same current flows through the resistor and the inductor. The voltages on the resistor and the inductor are $3 \cdot 0.5 \frac{d}{d t} v(t)$, and $1 \cdot 0.5 \frac{d^{2}}{d t^{2}} v(t)$ respectively. By employing the Kirchhoff's voltage law we have

$$
\begin{aligned}
& 0.5 \frac{d^{2}}{d t^{2}} v(t)+1.5 \frac{d}{d t} v(t)+v(t)=e(t), \quad \text { or } \\
& \frac{d^{2}}{d t^{2}} v(t)+3 \frac{d}{d t} v(t)+2 v(t) \quad=2 e(t)
\end{aligned}
$$

Response of the system $(h(t))$ to the unit impulse is needed. Because the system is casual $v(t)=0$, for $t<0$. We should find the system output for $t>0$. For $t>0$, the unit impulse is zero and hence the right side of the equation is zero. The unit impulse imposes initial values to the system. Therefore,

$$
\begin{aligned}
\frac{d^{2}}{d t^{2}} h(t)+3 \frac{d}{d t} h(t)+2 h(t) & =2 \delta(t) & & t \in \mathbb{R} \\
\frac{d^{2}}{d t^{2}} h(t)+3 \frac{d}{d t} h(t)+2 h(t) & =0 & & t>0
\end{aligned}
$$

In order to extract initial, $h\left(0^{+}\right)$, imposed by the impulse we integrate the differential equation (the first one in the above equations) twice over the interval $\left[0^{-}, 0^{+}\right]$;

$$
\begin{array}{ll}
\left.h(t)\right|_{0^{-}} ^{0^{+}}+\left.3 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}}+\left.2 \mathcal{D}^{-2} h(t)\right|_{0^{-}} ^{0^{+}} & =\left.2 t u(t)\right|_{0^{-}} ^{0^{+}} \\
h\left(0^{+}\right)-h\left(0^{-}\right)+3 \cdot 0+2 \cdot 0 & =0 \\
h\left(0^{+}\right) & =0
\end{array}
$$

To extract the second initial, $h^{\prime}\left(0^{+}\right)$, imposed by the impulse we integrate the differential
equation once over the interval $\left[0^{-}, 0^{+}\right]$;

$$
\begin{array}{ll}
\left.h^{\prime}(t)\right|_{0^{-}} ^{0^{+}}+\left.3 h(t)\right|_{0^{-}} ^{0^{+}}+\left.2 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}} & =\left.2 u(t)\right|_{0^{-}} ^{0^{+}} \\
h^{\prime}\left(0^{+}\right)-h^{\prime}\left(0^{-}\right)+3 \cdot 0+2 \cdot 0 & =2-0 \\
h^{\prime}\left(0^{+}\right) & =2
\end{array}
$$

Now, we are ready to solve the homogenous differential equation;

$$
\frac{d^{2}}{d t^{2}} h(t)+3 \frac{d}{d t} h(t)+2 h(t)=0
$$

The characteristic equation of this differential equation is

$$
D^{2}+3 D+2=0 \Rightarrow(D+2)(D+1)=0
$$

The roots of the characteristic equation are $D_{1}=-2$ and $D_{2}=-1$. Therefore the solution of the homogenous equation is

$$
h(t)=A e^{-2 t}+B e^{-t}
$$

The unknown constants, $A$ and $B$, are determined from the initial values.

$$
\begin{array}{ll}
h^{\prime}(t) & =-2 A e^{-2 t}-B e^{-t} \\
& \\
h^{\prime}\left(0^{+}\right) & =-2 A-B \\
h\left(0^{+}\right) & =A+B
\end{array}
$$

From the sum of these equations $A$ is obtained as $A=-2$. And by the second equation we get $B=2$. Consequently, the impulse response of the system

$$
h(t)=\left\{\begin{array}{ll}
0, & t<0 \\
-2 e^{-2 t}+2 e^{-t}, & t>0
\end{array}=-2 e^{-2 t} u(t)+2 e^{-t} u(t)\right.
$$

A4)

$$
x[n] * y[-n]=\sum_{k=-\infty}^{\infty} x[k] y[-(n-k)]=\sum_{k=-\infty}^{\infty} x[k] y[k-n]
$$

The sum, $\sum_{k=-\infty}^{\infty} x[k] y[k-n]$, is named as correlation of $x[n]$ with $y[n]$.

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} x[k] y[k-n] & =x[-1] y[-1-n]+x[0] y[-n]+x[1] y[1-n] \\
& =y[-1-n]+2 y[-n]+3 y[1-n]
\end{aligned}
$$

$$
\begin{array}{lrrrrrrrrl}
n & -2 & -1 & 0 & 1 & 2 & 3 & \text { otherwise } & & \\
x[n] & 0 & 1 & 2 & 3 & 0 & 0 & 0 & & \\
y[-1-n] & 3 & 2 & 1 & -1 & 0 & 0 & 0 & \times & 1 \\
y[-n] & 0 & 3 & 2 & 1 & -1 & 0 & 0 & \times & 2 \\
y[1-n] & 0 & 0 & 3 & 2 & 1 & -1 & 0 & \times & 3 \\
x[n] * y[-n] & 3 & 8 & 14 & 7 & 1 & -3 & 0 & &
\end{array}
$$




A5) The convolution integral

$$
z(t)=\int_{-\infty}^{\infty} x(\lambda) h(t-\lambda)
$$

$$
1+t<0 \quad \rightarrow \quad t<-1 \quad z(t)=0
$$

$$
t<0 \text { and } 1+t>0 \rightarrow 0>t>-1 \quad z(t)=\int_{0}^{1+t}\left(-\frac{1}{2} \lambda+1\right) d \lambda=-\frac{1}{4}(t+1)^{2}+t+1
$$

$$
t>0 \text { and } 1+t<2 \rightarrow 1>t>0 \quad z(t)=\int_{t}^{1+t}\left(-\frac{1}{2} \lambda+1\right) d \lambda=\frac{3}{4}-\frac{1}{2} t
$$

$$
t<2 \text { and } 1+t>2 \rightarrow 2>t>1 \quad z(t)=\int_{t}^{2}\left(-\frac{1}{2} \lambda+1\right) d \lambda=\frac{1}{4} t^{2}-t+1
$$

$$
t>2 \quad z(t)=0
$$

A6)
a)

$$
\int_{-\infty}^{\infty} \delta^{\prime}(t) d t=\lim _{\Delta \rightarrow 0} \int_{-\infty}^{\infty} a^{\prime}(t) d t=\lim _{\Delta \rightarrow 0} 0=0
$$

Alternatively,

$$
\int_{-\infty}^{\infty} \delta^{\prime}(t) d t=\left.\delta(t)\right|_{-\infty} ^{\infty}=\delta(\infty)-\delta(-\infty)=0
$$



b)

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{d}{d t}[\delta(t) g(t)] d t & =\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t+\int_{-\infty}^{\infty} \delta(t) g^{\prime}(t) d t \\
\left.\delta(t) g(t)\right|_{-\infty} ^{\infty} & =\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t+g^{\prime}(0) \\
\delta(\infty) g(\infty)-\delta(-\infty) g(-\infty) & =\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t+g^{\prime}(0) \\
& =\int_{-\infty}^{\infty} \delta^{\prime}(t) g(t) d t+g^{\prime}(0) \\
0 & =-g^{\prime}(0)
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
\int_{-\infty}^{\infty} g(t) a^{\prime}(t) d t & =\int_{-\Delta}^{0} g(t) \frac{1}{\Delta^{2}} d t+\int_{0}^{\Delta} g(t)\left(-\frac{1}{\Delta^{2}}\right) d t \\
& \approx g(-\Delta / 2) \frac{1}{\Delta^{2}} \Delta+g(\Delta / 2)\left(-\frac{1}{\Delta^{2}}\right) \Delta \\
& =-\frac{g(\Delta / 2)-g(-\Delta / 2)}{\Delta} \\
\lim _{\Delta \rightarrow 0} \int_{-\infty}^{\infty} g(t) a^{\prime}(t) d t & =-\lim _{\Delta \rightarrow 0} \frac{g(\Delta / 2)-g(-\Delta / 2)}{\Delta} \\
\int_{-\infty}^{\infty} g(t) \delta^{\prime}(t) d t & =-g^{\prime}(0)
\end{aligned}
$$

A7)

$$
y(t)+5 \mathcal{D}^{-1} y(t)+6 \mathcal{D}^{-2} y(t)=-\mathcal{D}^{-1} x(t)+2 \mathcal{D}^{-2} x(t)
$$

Direct form I :

$$
\begin{array}{ll}
w(t) & =-\mathcal{D}^{-1} x(t)+2 \mathcal{D}^{-2} x(t) \\
y(t)+5 \mathcal{D}^{-1} y(t)+6 \mathcal{D}^{-2} y(t) & =w(t)
\end{array}
$$

Direct form II :

$$
\begin{aligned}
v(t)+5 \mathcal{D}^{-1} v(t)+6 \mathcal{D}^{-2} v(t) & =x(t) \\
y(t) & =-\mathcal{D}^{-1} v(t)+2 \mathcal{D}^{-2} v(t)
\end{aligned}
$$



A8)

$$
\begin{array}{ll}
h^{\prime \prime}(t)+5 h^{\prime}(t)+6 h(t)=-\delta^{\prime}(t)+2 \delta(t) & \\
t \in \mathbb{R} \\
h^{\prime \prime}(t)+5 h^{\prime}(t)+6 h(t)=0 & t>0
\end{array}
$$

In order to extract initial, $h\left(0^{+}\right)$, imposed by the impulse we integrate the first differential equation in the above equations twice over the interval $\left[0^{-}, 0^{+}\right]$;

$$
\begin{aligned}
\left.h(t)\right|_{0^{-}} ^{0^{+}}+\left.5 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}}+\left.6 \mathcal{D}^{-2} h(t)\right|_{0^{-}} ^{0^{+}} & =-\left.u(t)\right|_{0^{-}} ^{0^{+}}+\left.2 t u(t)\right|_{0^{-}} ^{0^{+}} \\
h\left(0^{+}\right)-h\left(0^{-}\right)+5 \cdot 0+6 \cdot 0 & =-1+0+0 \\
h\left(0^{+}\right) & =-1
\end{aligned}
$$

To extract the second initial, $h^{\prime}\left(0^{+}\right)$, imposed by the impulse we integrate the differential equation once over the interval $\left[0^{-}, 0^{+}\right]$;

$$
\begin{array}{ll}
\left.h^{\prime}(t)\right|_{0^{-}} ^{0^{+}}+\left.5 h(t)\right|_{0^{-}} ^{0^{+}}+\left.6 \mathcal{D}^{-1} h(t)\right|_{0^{-}} ^{0^{+}} & =-\left.\delta(t)\right|_{0^{-}} ^{0^{+}}+\left.2 u(t)\right|_{0^{-}} ^{0^{+}} \\
h^{\prime}\left(0^{+}\right)-h^{\prime}\left(0^{-}\right)+5 \cdot(-1)+6 \cdot 0 & =-\delta\left(0^{+}\right)+\delta\left(0^{-}\right)+2 u\left(0^{+}\right)-2 u\left(0^{-}\right) \\
h^{\prime}\left(0^{+}\right)-0-5+0 & =-0+0+2-0 \\
h^{\prime}\left(0^{+}\right) & =7
\end{array}
$$

The characteristic equation of this differential equation is

$$
D^{2}+5 D+6=0 \Rightarrow(D+2)(D+3)=0
$$

The roots of the characteristic equation are $D_{1}=-2$ and $D_{2}=-3$. Therefore the solution of the homogenous equation is

$$
h(t)=A e^{-2 t}+B e^{-3 t}
$$

The unkown constants, $A$ and $B$, are determined from the initial values.

$$
\begin{array}{rlrl}
h^{\prime}(t) & =-2 A e^{-2 t}-3 B e^{-3 t} & & \\
h^{\prime}\left(0^{+}\right) & =-2 A-3 B & =7 \\
h\left(0^{+}\right) & =A+B & =-1
\end{array}
$$

Multiplying the second equation by 3 and then adding these equations $A$ is obtained as $A=4$. And by the second equation we get $B=-5$. Consequently, the impulse response of the system

$$
h(t)=\left\{\begin{array}{ll}
0, & t<0 \\
4 e^{-2 t}-5 e^{-3 t}, & t>0
\end{array}=4 e^{-2 t} u(t)-5 e^{-3 t} u(t)\right.
$$

A9) The output is -1 Volt just before switched is closed. When the input is not an impulse the output (here the capacitor voltage) does change its value immediately; it is continuous. Therefore $y\left(0^{+}\right) \approx y\left(0^{-}\right)=-1$. In the linearity test we use DC inputs. Then, consider that the input is $A$ Volts. The input-output relation of this circuit simply a first order differential equation for $t>0$.

$$
y^{\prime}(t)+y(t)=x(t)
$$

The solution when $x(t)=A$ is

$$
y(t)=-(A+1) e^{-t}+A, \quad t>0
$$

In the table below we test additivity and homogeneity properties.

$$
\begin{array}{lll}
\text { Input } & \text { Output } & \\
x_{1}(t)=1 & y_{1}(t)=-2 e^{-t}+1, & t>0 \\
x_{2}(t)=2 & y_{2}(t)=-3 e^{-t}+2, & t>0 \\
x_{3}(t)=x_{1}(t)+x_{2}(t)=3 & y_{3}(t)=-4 e^{-t}+3, & t>0 \\
x_{2}(t)=2 x_{1}(t)=2 & y_{3}(t) \neq y_{1}(t)+y_{2}(t) & \\
& y_{2}(t)=-3 e^{-t}+2, \quad t>0 \\
y_{2}(t) \neq 2 y_{1}(t) & & \\
& \text { additivity is not satisfied } \\
\text { homogeneity is not satisfied }
\end{array}
$$

Both, additivity and homogeneity are not satisfied. This system is not linear. But, if the initial was zero the linearity would be satisfied. Therefore the system is incrementally linear.

In general, the output of an incrementally linear system for an input $x(t)$ excerted at time $t=t_{0}$ is in the form

$$
y(t)=\int_{t_{0}^{+}}^{t} x(\lambda) h(t-\lambda) d \lambda+y\left(t_{0}^{+}\right), \quad t>t_{0}
$$

where $h(t)$ is the impulse response when all initials are zero. As one can easly observe that integral in the equation is a linear operator and that the initial value of the output is the same for any input. If the initial value $y\left(t_{0}^{+}\right)$is zero the system is linear. Consequently, linearity of a system is violated when initials are not zero.

A10)




