Instructions Answer all questions. Give your answers clearly. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Calculator and cell phone are not allowed in the exam. Each question is worth 12 points. **Time** 100 minutes.

1	2	3	4	5	6	7	8	9	10	TOTAL

QUESTIONS

Q1) Find impulse response of the following LTI-causal system in time-domain (do not employ any transform method).

$$y''(t) + 2y'(t) = x(t)$$

Q2) Find impulse response of the following LTI-causal system time-domain (do not employ any transform method).

$$y[n] - \frac{1}{9}y[n-2] = x[n]$$

Q3) Frequency response of an LTI-causal system is given as

$$H(\omega) = \frac{1}{j\omega + 2} + \frac{1}{j\omega + 1}$$

Find the corresponding differential equation which characterize the LTI system.

Q4) Compute Laplace transform of the following signal. Determine and plot region of converge of the Laplace transform.

$$x(t) = 3e^{-2|t|} = 3e^{2t}u(-t) + 3e^{-2t}u(t)$$

You can employ

$$\mathcal{L}[\,e^{-at}u(t)\,] = \frac{1}{s+a}, \quad \mathrm{Re}[\,s\,] > -a \quad \text{and} \quad \mathcal{L}[\,-e^{-at}u(-t)\,] = \frac{1}{s+a}, \quad \mathrm{Re}[\,s\,] < -a$$

Q5) An eigen-function $x(t) = 3e^{-2t}$ is exerted to the input of an LTI-system with an impulse response $h(t) = e^{-3t}u(t)$. Compute the eigen-value corresponding to this eigen-function. What is the output for this input?
You can make use of

$$\mathcal{L}\left[\,e^{-at}u(t)\,\right] = \frac{1}{s+a}, \quad \mathrm{Re}[\,s\,] > -a$$

Q6) Frequency response of an LTI system is given as in the following. Find magnitude and phase of the frequency response. Note that the phase of complex function $z(\omega) = 0 + j\omega$ is discontinuous at $\omega = 0$.

$$H(\omega) = \frac{j\omega}{j\omega + 2}$$

Q7) Compute complex Fourier series coefficients of the following periodic signal.

$$x(t) = 4\cos^2\left(\frac{\pi}{6}t\right) - 8\sin\left(\frac{2\pi}{9}t\right)$$

You can apply Euler's identity: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Q8) Find trigonometric Fourier series representation of the signal, $x(t) = |\sin(\pi t)|$.

Q9) Consider that $x(t)$ is a periodic signal and its complex Fourier series coefficient is a_k. Another signal is derived from this signal by $y(t) = x(\alpha t)$ and let complex Fourier series coefficient of the derived signal is b_k. Find the relation between b_k and a_k for $\alpha > 0$ and $\alpha < 0$.

Q10) A signal

$$\begin{aligned} x(t) &= u(t+2) - 2u(t) + u(t-2) \\ &= \begin{cases} 1, & -2 < t < 0 \\ -1, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

is given.

a) Plot this signal. b) Find and plot

$$y(t) = \int_{-\infty}^{t} x(2\lambda/3 + 1)\, d\lambda = \frac{3}{2} \int_{-\infty}^{2t/3+1} x(\lambda)\, d\lambda$$

ANSWERS

A1) The initials can be easly deduced from the differential equation by integrating it ones and twice over the interval $\left[0^-, 0^+\right]$. The initials are obtained as $h\left(0^+\right) = 0$ and $h'\left(0^+\right) = 1$. The characteristic equation is

$$D^2 + 2D = 0$$

and the roots of this equation are $D_1 = 0$ and $D_2 = 1$. Therefore the impulse response for $t > 0$ is

$$h(t) = A + Be^{-2t}$$

We employ initials to compute the unknown coefficients A and B.

$$\begin{aligned} h(t) &= A + Be^{-2t} \qquad & A + B &= 0 \\ h'(t) &= -2Be^{-2t} \qquad & -2B &= 1 \end{aligned}$$

Then, $B = -1/2$ and $A = 1/2$. As a result,

$$h(t) = \frac{1}{2}\left(1 - e^{-2t}\right)u(t)$$

A2) The initials are extracted as $h[0] = 1$ and $h[1] = 0$ by computing the difference equation at time $n = 0$ and $n = 1$. The characteristic equation is

$$1 - \frac{1}{9}D^{-2} = 0$$

and the roots of this equation are $D_1 = -\frac{1}{3}$ and $D_2 = -\frac{1}{3}$. Therefore the impulse response for $n > -1$ is

$$h[n] = A\left(-\frac{1}{3}\right)^n + B\left(\frac{1}{3}\right)^n$$

We employ initials to compute the unknown coefficients A and B.

$$\begin{aligned} A + B &= 1 \\ -\frac{1}{3}A + \frac{1}{3}B &= 0 \end{aligned}$$

We extract the coefficients as, $A = 1/2$ and $B = 1/2$. Consequently,

$$h[n] = \frac{1}{2}\left(-\frac{1}{3}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{3}\right)^n u[n] = \begin{cases} \left(\frac{1}{3}\right)^n, & n = 0, 2, 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

A3)

$$H(\omega) = \frac{1}{j\omega + 2} + \frac{1}{j\omega + 1} = \frac{2j\omega + 3}{(j\omega)^2 + 3j\omega + 2} = \frac{Y(\omega)}{X(\omega)}$$

$$\begin{aligned} \left((j\omega)^2 + 3j\omega + 2\right)Y(\omega) &= (2j\omega + 3)X(\omega) \\ (j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) &= 2j\omega X(\omega) + 3X(\omega) \end{aligned}$$

$$\begin{aligned} \mathcal{F}^{-1}\left[(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega)\right] &= \mathcal{F}^{-1}\left[2j\omega X(\omega) + 3X(\omega)\right] \\ \mathcal{F}^{-1}\left[(j\omega)^2 Y(\omega)\right] + 3\mathcal{F}^{-1}\left[j\omega Y(\omega)\right] + 2\mathcal{F}^{-1}\left[Y(\omega)\right] &= 2\mathcal{F}^{-1}\left[j\omega X(\omega)\right] + 3\mathcal{F}^{-1}\left[X(\omega)\right] \\ \frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(\omega) &= \frac{d}{dt}x(t) + 3x(t) \end{aligned}$$

A4)

$$\begin{aligned}
\mathcal{L}\left[\,x(t)\,\right] &= \mathcal{L}\left[\,3e^{2t}u(-t)+3e^{-2t}u(t)\,\right] \\
X(s) &= 3\mathcal{L}\left[\,e^{2t}u(-t)\,\right]+3\mathcal{L}\left[\,e^{-2t}u(t)\,\right] \\
&= -\frac{3}{s-2}+\frac{3}{s+2}, \qquad (\mathrm{Re}\,[\,s\,]<2)\cap(\mathrm{Re}\,[\,s\,]>-2) \\
&= -\frac{12}{s^2-4}, \qquad -2<\mathrm{Re}\,[\,s\,]<2
\end{aligned}$$

A5) The eigen-vaue of the system for the eigen-function, e^{st}, is

$$\begin{aligned}
H(s) &= \mathcal{L}\left[\,h(t)\,\right] = \mathcal{L}\left[\,e^{-3t}u(-t)\,\right] \\
&= \frac{1}{s+3}, \qquad \mathrm{Re}\,[\,s\,]>-3
\end{aligned}$$

Therefore, for $x(t)=3e^{-2t}$, the eigen-value is $H(-2)=1$. And the output is $y(t)=3H(-2)\,e^{-2t}=3e^{-2t}$.

A6) Polar form of the complex function $H(\omega)$ is $H(\omega)=A(\omega)\,e^{j\theta(\omega)}$.

$$\begin{aligned}
A(\omega) &= \sqrt{\frac{\omega^2}{\omega^2+4}} = \sqrt{\frac{1}{1+\frac{4}{\omega^2}}} \\
\theta(\omega) &= \arctan\frac{\omega}{0}-\arctan\frac{\omega}{2} = \begin{cases} -\frac{\pi}{2}-\arctan\frac{\omega}{2}, & \omega<0 \\ \frac{\pi}{2}-\arctan\frac{\omega}{2}, & \omega>0 \end{cases}
\end{aligned}$$

A7) Using the trigonometric identity; $\cos^2(\theta)=\frac{1}{2}+\frac{1}{2}\cos(2\theta)$ and the equalities; $\cos(\theta)=\frac{1}{2}e^{j\theta}+\frac{1}{2}e^{-j\theta}$ and $\sin(\theta)=\frac{1}{2j}e^{j\theta}-\frac{1}{2j}e^{-j\theta}$ we can easly extract complex Fourier series coefficients. The period of the signal is $\mathrm{lcm}(6,9)=18$.

$$\begin{aligned}
x(t) &= 4\cos^2\left(\frac{\pi}{6}t\right)-8\sin\left(\frac{2\pi}{9}t\right) \\
&= 2+2\cos\left(\frac{\pi}{3}t\right)-8\sin\left(\frac{2\pi}{9}t\right) \\
&= 2+e^{j\pi/3t}+e^{-j\pi/3t}+4j\,e^{j2\pi/9t}-4j\,e^{-j2\pi/9t} \\
&= 2+e^{j(2\pi/18)\cdot 3\cdot t}+e^{j(2\pi/18)\cdot(-3)\cdot t}+4j\,e^{j(2\pi/18)\cdot 2\cdot t}-4j\,e^{j(2\pi/18)\cdot(-2)\cdot t} \\
&= a_0+a_3e^{j(2\pi/18)\cdot 3\cdot t}+a_{-3}e^{j(2\pi/18)\cdot(-3)\cdot t}+a_2e^{j(2\pi/18)\cdot 2\cdot t}+a_{-2}\,e^{j(2\pi/18)\cdot(-2)\cdot t}
\end{aligned}$$

Then,the coefficients are

$$\begin{aligned}
a_{-3} &= 1 \qquad a_{-2} = -4j \\
a_0 &= 2 \qquad a_2 = 4j \\
a_3 &= 1 \qquad a_k = 0, \qquad k\neq -3,-2,0,2,3
\end{aligned}$$

A8) The fundemental period of $\sin(\pi t)$ is 2 seconds. The fundemental period of absolute value of $\sin(\pi t)$; $|\sin(\pi t)|$ becomes half of 2 seconds (1 second). Therefore, $T = 1$. And since the signal $x(t)$ is an even function, the trigonometric Fourier series does not contain odd terms; the coefficients of sin terms are zero.

$$\begin{aligned} a_0 &= \frac{1}{1}\int_0^1 \sin(\pi t)\,dt &&= -\frac{1}{\pi}\cos(\pi t)\Big|_0^1 \\ &= -\frac{1}{\pi}\cos(\pi) + \frac{1}{\pi}\cos(0) &&= \frac{2}{\pi} \end{aligned}$$

$$\begin{aligned} d_k &= \frac{2}{1}\int_0^1 \sin(\pi t)\cos(2\pi k t)\,dt \\ &= 2\int_0^1 \sin((1+2k)\pi t)\,dt + 2\int_0^1 \sin((1-2k)\pi t)\,dt \\ &= -\frac{2}{(1+2k)\pi}\cos((1+2k)\pi t)\Big|_0^1 - \frac{2}{(1-2k)\pi}\cos((1-2k)\pi t)\Big|_0^1 \\ &= -\frac{2}{(1+2k)\pi}\cos((1+2k)\pi) + \frac{2}{(1+2k)\pi}\cos(0) \\ &\quad -\frac{2}{(1-2k)\pi}\cos((1-2k)\pi) + \frac{2}{(1-2k)\pi}\cos(0) \\ &= \frac{4}{(1+2k)\pi} + \frac{4}{(1-2k)\pi} \\ &= \frac{8}{(1-4k^2)\pi} \end{aligned}$$

$$x(t) = |\sin(\pi t)| = \frac{2}{\pi} + \frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{(1-4k^2)}\cos(2\pi k t)$$

A9) A periodic signal with a period T can be expressed by complex Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

Then,

$$y(t) = x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}k\alpha t} = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{(T/\alpha)}kt}$$

The period of $y(t)$ is T/α for $\alpha > 0$, $-T/\alpha$ when $\alpha < 0$. Therefore, $b_k = a_k$ for $\alpha > 0$. Concerning, $\alpha < 0$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}k\alpha t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{-j\frac{2\pi}{T}k\alpha t} = \sum_{k=-\infty}^{\infty} a_{-k} e^{j\frac{2\pi}{(-T/\alpha)}kt}$$

We get, $b_k = a_{-k}$ for $\alpha > 0$.

A10)

$$g(t) = \int_{-\infty}^{t} x(\lambda)\,d\lambda, \qquad y(t) = \frac{3}{2}g(2t/3+1), \qquad y\left(\frac{3}{2}t - \frac{3}{2}\right) = \frac{3}{2}g(t)$$

$$g(t) = \begin{cases} 2+t, & -2 < t < 0 \\ 2-t, & 0 \leq t < 2 \\ 0, & \text{otherwise} \end{cases} \qquad y(t) = \begin{cases} \frac{9}{2}+t, & -\frac{9}{2} < t < -\frac{3}{2} \\ \frac{3}{2}-t, & -\frac{3}{2} \leq t < \frac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$