**Instructions** Answer all questions. Give your answers clearly. Do not leave mathematical operations incomplete (do not skip intermediate operations and obtain the possible simplest form of the results). Calculator and cell phone are not allowed in the exam. Each question is worth 20 points. **Time** 120 minutes.



## QUESTIONS

Q1) Find and plot y(t) = x(-t/2+1) - x(t/2+1) for the following signal.



- Q2) Find the convolution;  $x(t) = 2\cos(t) * e^{-t}u(t)$ .
- Q3) a) Compute Fourier transform of  $x(t) = 2\cos(t)u(t) * e^{-t}u(t)$ . b) Obtain inverse Fourier transform of

$$X(\omega) = \frac{1}{(j\omega+2)\left[(j\omega)^2 + 2j\omega + 1\right]}$$

Q4) Find a) the frequency response, b) the impulse response of the following LTI-causal system.

$$y''(t) + 2y'(t) + 5y(t) = x(t) - x'(t)$$

Q5) Find (bi-lateral) Laplace transform of the signal;  $x(t) = 2e^{-t} \cos(t) u(t) + 3e^{2t} u(-t)$  (specify and show the region of the convergence in the complex plane).

Q6) A series RLC circuit is given in the following. Consider that the state-variables for this circuit is the inductor current  $i_L(t)$  and the capacitor voltage  $v_C(t)$ . Obtain the state-form of the input-output relation (the matrix-form of the state-equations) of this system.

Follow these steps:

a) Find  $v'_{C}(t)$  from the current-voltage relation of the capacitor.

b) Extract  $i'_{L}(t)$  using Kirchhoff's voltage law.

c) The output is easily is obtained from the sate-variables by employing Kirchhoff's voltage law:  $y(t) = x(t) - Ri_L(t)$ .



Q6

## ANSWERS

A1)



A2)

$$e^{j\omega_{0}t} * h(t) = e^{j\omega_{0}t} \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_{0}\lambda} d\lambda = e^{j\omega_{0}t} H(\omega_{0})$$
  

$$x(t) = 2\cos(t) * e^{-t}u(t) = (e^{jt} + e^{-jt}) * e^{-t}u(t)$$
  

$$= e^{jt} * e^{-t}u(t) + e^{-jt} * e^{-t}u(t)$$
  

$$= e^{jt} H(1) + e^{-jt} H(-1)$$

## $H(\omega) = \mathcal{F}\left[e^{-t}u(t)\right] = \frac{1}{1+j\omega}$ $H(1) = \frac{1}{1+j}$ $H(-1) = \frac{1}{1-j}$ $x(t) = e^{jt}\frac{1}{1+j} + e^{-jt}\frac{1}{1-j}$ $= e^{jt}\frac{1-j}{2} + e^{-jt}\frac{1+j}{2}$ $= \frac{1}{2}\left[e^{jt} + e^{-jt}\right] - \frac{1}{2}j\left[e^{jt} - e^{-jt}\right]$ $= \cos(t) + \sin(t)$

A3) a) From the convolution property of the Fourier transform we can write  $X(\omega) = 2\mathcal{F}[\cos(t) u(t)] \cdot \mathcal{F}[e^{-t}u(t)].$ 

$$\mathcal{F}\left[\cos\left(\omega_{0}t\right)u\left(t\right)\right] = \frac{\pi}{2}\left[\delta\left(\omega-\omega_{0}\right)+\delta\left(\omega+\omega_{0}\right)\right]+\frac{j\omega}{\omega_{0}^{2}-\omega^{2}}$$
$$\mathcal{F}\left[\cos\left(t\right)u\left(t\right)\right] = \frac{\pi}{2}\left[\delta\left(\omega-1\right)+\delta\left(\omega+1\right)\right]+\frac{j\omega}{1-\omega^{2}}$$
$$\mathcal{F}\left[e^{-at}u\left(t\right)\right] = \frac{1}{a+j\omega}$$
$$\mathcal{F}\left[e^{-t}u\left(t\right)\right] = \frac{1}{1+j\omega}$$
$$X\left(\omega\right) = 2\cdot\left[\frac{\pi}{2}\delta\left(\omega-1\right)+\frac{\pi}{2}\delta\left(\omega+1\right)+\frac{j\omega}{1-\omega^{2}}\right]\cdot\frac{1}{1+j\omega}$$
$$= \frac{\pi}{1+j}\delta\left(\omega-1\right)+\frac{\pi}{1-j}\delta\left(\omega+1\right)+\frac{j\omega}{1-\omega^{2}}\frac{1}{1+j\omega}$$

b)

$$X(\omega) = \frac{1}{(j\omega+2)\left[(j\omega)^2 + 2j\omega+1\right]}$$
$$= \frac{1}{(j\omega+2)(j\omega+1)^2}$$
$$= \frac{A}{j\omega+2} + \frac{B}{j\omega+1} + \frac{C}{(j\omega+1)^2}$$

$$A = \lim_{j\omega\to-2} (j\omega+2) X(\omega) = \lim_{j\omega\to-2} \frac{1}{(j\omega+1)^2}$$
  
=  $\frac{1}{(-2+1)^2}$  = 1  
$$C = \lim_{j\omega\to-1} (j\omega+1)^2 X(\omega) = \lim_{j\omega\to-1} \frac{1}{j\omega+2}$$
  
=  $\frac{1}{-1+2}$  = 1  
$$B = \lim_{j\omega\to-1} \frac{d}{d(j\omega)} (j\omega+1)^2 X(\omega) = \lim_{j\omega\to-1} -\frac{1}{(j\omega+2)^2}$$
  
=  $-\frac{1}{(-1+2)^2}$  = -1

$$\begin{split} X\left(\omega\right) &= \frac{1}{j\omega+2} - \frac{1}{j\omega+1} + \frac{1}{\left(j\omega+1\right)^2} \\ \mathcal{F}^{-1}\left[X\left(\omega\right)\right] &= \mathcal{F}^{-1}\left[\frac{1}{j\omega+2}\right] - \mathcal{F}^{-1}\left[\frac{1}{j\omega+1}\right] + \mathcal{F}^{-1}\left[\frac{1}{\left(j\omega+1\right)^2}\right] \\ x\left(t\right) &= e^{-2t}u\left(t\right) - e^{-t}u\left(t\right) + te^{-t}u\left(t\right) \end{split}$$

A4) a)

$$y''(t) + 2y'(t) + 5y(t) = x(t) - x'(t)$$

$$\mathcal{F}[y''(t) + 2y'(t) + 5y(t)] = \mathcal{F}[x(t) - x'(t)]$$

$$\mathcal{F}[y''(t)] + 2\mathcal{F}[y'(t)] + 5\mathcal{F}[y(t)] = \mathcal{F}[x(t)] - \mathcal{F}[x'(t)]$$

$$(j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + 5Y(\omega) = X(\omega) - j\omega X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$= \frac{1 - j\omega}{(j\omega)^2 + 2j\omega + 5}$$

b)

$$H(\omega) = \frac{1-j\omega}{(j\omega)^2 + 2j\omega + 5}$$
$$= \frac{1-j\omega}{(j\omega + 1 - 2j)(j\omega + 1 + 2j)}$$
$$= \frac{A}{j\omega + 1 - 2j} + \frac{B}{j\omega + 1 + 2j}$$

$$A = \lim_{j\omega \to -1+2j} (j\omega + 1 - 2j) H(\omega) = \lim_{j\omega \to -1+2j} \frac{1 - j\omega}{j\omega + 1 + 2j}$$
$$= \frac{1 - (-1 + 2j)}{-1 + 2j + 1 + 2j} = \frac{2 - 2j}{4j}$$
$$= -\frac{1}{2} - \frac{1}{2}j$$

$$B = \lim_{j\omega \to -1-2j} (j\omega + 1 + 2j) H(\omega) = \lim_{j\omega \to -1-2j} \frac{1 - j\omega}{j\omega + 1 - 2j}$$
$$= \frac{1 - (-1 - 2j)}{-1 - 2j + 1 - 2j} = \frac{2 + 2j}{-4j}$$
$$= -\frac{1}{2} + \frac{1}{2}j$$

$$\begin{split} h(t) &= \mathcal{F}^{-1} \left[ H(\omega) \right] \\ &= \mathcal{F}^{-1} \left[ \frac{A}{j\omega + 1 - 2j} + \frac{B}{j\omega + 1 + 2j} \right] \\ &= A \mathcal{F}^{-1} \left[ \frac{1}{j\omega + 1 - 2j} \right] + B \mathcal{F}^{-1} \left[ \frac{1}{j\omega + 1 + 2j} \right] \\ &= A e^{-(1 - 2j)t} u(t) + B e^{-(1 + 2j)t} u(t) \\ &= -\left( \frac{1}{2} + \frac{1}{2}j \right) e^{-(1 - 2j)t} u(t) - \left( \frac{1}{2} - \frac{1}{2}j \right) e^{-(1 + 2j)t} u(t) \\ &= -e^{-t} u(t) \frac{1}{2} \left[ e^{2jt} + e^{-2jt} \right] + e^{-t} u(t) \frac{1}{2j} \left[ e^{2jt} - e^{-2jt} \right] \\ &= e^{-t} \left[ \sin(2t) - \cos(2t) \right] u(t) \end{split}$$

$$\begin{split} \text{A5)} \ x \left( t \right) &= 2e^{-t} \cos \left( t \right) u \left( t \right) + 3e^{2t} u \left( -t \right) \\ \mathcal{L} \left[ e^{-at} \cos \left( bt \right) u \left( t \right) \right] &= \frac{s+a}{(s+a)^2 + b^2}, & \text{Re} \left[ s \right] > -a \\ \mathcal{L} \left[ e^{-t} \cos \left( t \right) u \left( t \right) \right] &= \frac{s+1}{(s+1)^2 + 1}, & \text{Re} \left[ s \right] > -1 \\ \mathcal{L} \left[ -e^{-at} u \left( -t \right) \right] &= \frac{1}{s+a}, & \text{Re} \left[ s \right] < -a \\ \mathcal{L} \left[ e^{2t} u \left( -t \right) \right] &= -\frac{1}{s-2}, & \text{Re} \left[ s \right] < 2 \\ X \left( s \right) &= 2\frac{s+1}{(s+1)^2 + 1} - 3\frac{1}{s-2}, & -1 < \text{Re} \left[ s \right] < 2 \\ &= -\frac{s^2 + 8s + 10}{(s^2 + 2s + 1) \left( s - 2 \right)} \\ &= -\frac{\left( s + 4 + \sqrt{6} \right) \left( s + 4 - \sqrt{6} \right)}{(s^2 + 2s + 1) \left( s - 2 \right)}. \end{split}$$



A5. The region of convergence.

A6) a)

$$i_{L}(t) = \frac{1}{4} \frac{d}{dt} v_{C}(t)$$
  

$$v'_{C}(t) = 4i_{L}(t)$$
  
b)  

$$-x(t) + 5i_{L}(t) + 1 \cdot \frac{d}{dt} i_{L}(t) + v_{c}(t) = 0$$
  

$$-v_{c}(t) - 5i_{L}(t) + x(t) = i'_{L}(t)$$

c)

$$y(t) = x(t) - 5i_L(t)$$

The state-form is as follows.

$$\begin{bmatrix} v'_{C}(t) \\ i'_{L}(t) \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -1 & -5 \end{bmatrix} \cdot \begin{bmatrix} v_{C}(t) \\ i_{L}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot x(t)$$

$$y(t) = \begin{bmatrix} 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} v_{C}(t) \\ i_{L}(t) \end{bmatrix} + x(t)$$